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A SPATIAL MARKOV QUEUEING PROCESS AND ITS APPLICATIONS TO WIRELESS LOSS SYSTEMS

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Abstract

We consider a pure-jump Markov generator which can be seen as a generalization of the spatial birth-and-death generator, which allows for mobility of particles. Conditions for the regularity of this generator and for its ergodicity are established. We also give the conditions under which its stationary distribution is a Gibbs measure. This extends previous work in [13] by allowing particle mobility. Such spatial birth-mobility-and-death processes can also be seen as generalizations of the spatial queueing systems considered in [15]. So our approach yields regularity conditions and alternative conditions for ergodicity of spatial open Whittle networks, complementing the results in [10]. Next we show how our results can be used to model wireless communication networks. In particular we study two spatial loss models for which we establish an expression for the blocking probability that might be seen as a spatial version of the classical Erlang loss formula. Some specific applications to CDMA (Code Division Multiple Access) networks are also discussed.

Keywords: spatial birth-and-death process, spatial queueing network, mobility, regularity, ergodicity, invariant measure, Gibbs measure, spatial Erlang formula

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1. Introduction

Object of interest. In this paper, we introduce the class of *Spatial Markov Queueing* (SMQ) processes, which are pure jump Markov processes which take their values in the space of finite counting measures on some general, complete, separable, metric space. We think of these counting measures as describing locations of some *users* in the space. The SMQ process evolves because of users being born, moving or dying, with only one such event being possible at a time. The process is defined by its generator, which describes the behavior of each user by a common, fixed *Markov routing kernel* and *arrival-departure rates* which represent the intensities with which each user is "repulsed" from its present location and is "attracted" to a new one. These rates possibly depend on the entire configuration of users. Some special cases

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of SMQ processes are *Spatial Birth-and-Death* (SBD) processes, where users do not move, and *Markov Poisson Location* (MPL) processes where users are born, move and die independently of each other. Spatial queueing Jackson and Whittle networks are special cases too.

Main goals. The main goal of this paper is to give sufficient conditions for the SMQ generator to be *regular* and *ergodic*. We also give conditions for its invariant measure to be *Gibbs*.

Mathematical techniques. We prove both regularity and ergodicity by comparing our SMQ process to a discrete birth-and-death process, for which the conditions for theses two properties are known (and given in [14] and in [11]). More specifically, in order to prove ergodicity, we use a dominating birth-and-death process to give sufficient conditions for the null measure (representing the empty-system) to be an ergodic state of the process. Then the limiting and invariant measure is given by the classical cycle formula. In some special cases, mostly when the process is reversible, this invariant measure is a Gibbs measure with respect to some Poisson point process with finite intensity measure.

Motivations. Our research is motivated by the analysis of wireless communications. Classical circuit switched loss models (with a discrete phase space) are well adapted to the analysis of wired communication networks, where the spatial components of the model are typically represented by some *graph of links*, and where the coexistence of calls on a common link is only possible if sufficiently many free resources are available on this link. In wireless communications, one needs to take into account the spatial characteristics of the network in a more thorough way because the relative locations of all radio channels determine their joint feasibility/rate. This is especially important for so called *interference limited systems* such as e.g. Code Division Multiple Access (CDMA).

We can roughly distinguish two types of applications of such networks.

- *Real-time applications* (voice calls, real-time audio-video streaming), which require a fixed minimal required bit-rate on each link, and which are momentarily denied access if it is not possible for the network to provide this bit rate. In order to analyze such services, one has to construct loss models and to study their loss and cut probabilities.
- *Elastic applications* (data traffic), which can momentarily cope with arbitrarily low bit-rate or large delays. In order to analyze such applications one typically uses queuing models and studies their waiting or sojourn times.

As we shall see, SMQ processes allow one to model both loss systems and queueing systems, and hence both types of applications within the spatial framework alluded to above. They also allow one to consider elastic traffic with guaranteed minimal bit-rates. Finally, they allow the representation of user mobility.

For all these reasons, such processes are more suitable for modeling of wireless communication systems than other previously studied spatial models such as SBD processes and spatial Jackson or Whittle networks.

To make this claim more clear, in the second part of the paper, we will use SMQ processes to define and analyze two loss models which cannot be seen as Whittle networks.

A concrete case. These loss models will be used to analyze CDMA wireless networks. CDMA is a medium access technique used in modern cellular networks (as e.g. UMTS). In CDMA, a given configuration of channels with predefined bit-rates is feasible if there exists some vector of transmitted powers which guarantee that the Signal-to-Interference-and-Noise-Ratio (SINR) at each receiver exceeds the threshold required for the bit-rate of the associated channel. A new mathematical representation of interferences based on shot noise has already led (see [2, 5]) to a variety of results on coverage and capacity of large CDMA networks. It allowed also for the definition of new decentralized admission and congestion control protocols. (see [6, 3]) In [4] blocking rates, defined as the fraction of users that are rejected by the admission control policy in the long run, were studied using SBD processes (i.e., without taking into account the mobility of users). In particular, a spatial Erlang formula was established. This formula connects the blocking rate and the infeasibility probability, defined as the steady state probability for the free SBD process of users to exceed what can be admitted by the system. In this paper we extend these results to general SMQ free processes and we study motion cut probabilities.

Related work. Some special cases of SMQ processes have already been studied. In particular SBD processes were treated in [13] and spatial Markov queueing processes (basically Whittle networks) in [10] (see also [15]). Our approach is inspired by [13] and is different from that of [10]. In the latter, the Gibbs invariant measure is first identified (at least the solution of the traffic equations) and then sufficient conditions for the null measure to be an ergodic state are found by comparing the process to a $M/G/\infty$ queueing system, via a MPL process with some modified system of traffic equations. Our method is more general and our sufficient conditions for ergodicity seem to be less constraining in cases when both approaches can be applied. In particular we do not need uniformly bounded arrival rates. The stability of some general (non necessarily Markov) spatial queueing systems, where the users are motionless, was also studied in [7].

Organization. The paper is organized as follows. In Section 2 we introduce notation and recall some basic facts concerning point processes and measure valued pure-jump Markov processes. In Section 3 we introduce our SMQ generator and give sufficient conditions for it to be regular and ergodic, as well as conditions for its invariant measure to be Gibbs. Next, in Section 4 we use the SMQ process to model two loss systems and we give formulas for blocking and cut probabilities. The CDMA case is studied in Section 5. The appendix contains some results which make the paper more self-contained.

2. Preliminaries

2.1. Point process

Very much as in [15], we will consider a system in which particles (users) live in a complete, separable metric space $\mathbb{D}$ with its Borel σ-field $\mathcal{D}$. Typically $\mathbb{D}$ would be a subset of some Euclidean space. If $\mathbb{D}$ is a finite set of points, the system is discrete. In the general case, we will represent the state of the system by a finite *counting measure*

ν on $\mathbb{D}$ defined by

$$\nu(A) = \sum_{i=1}^{\nu(\mathbb{D})} \varepsilon_{x_i}(A)$$

for $A \in \mathcal{D}$, where ε_x is a Dirac measure with unit mass at x (i.e. $\varepsilon_x(A) = 1$ if $x \in A$ and 0 otherwise) and where $x_1, \ldots, x_k \in \mathbb{D}$ are the locations of the particles. As a simple consequence of this notation we have that for all real valued measurable functions f on $\mathbb{D}$, $\int f(x)\, \nu(\mathrm{d}x) = \sum_{i=1}^{k} f(x_i)$.

A random configuration N of particles will be modeled by a random *point process* that is a measurable mapping from some given probability space to the state space $\mathbb{M}$ of all locally finite (finite on convex sets) counting measures on $\mathbb{D}$ (with the smallest σ-algebra $\mathcal{M}$ making the mappings $\mathbb{M} \ni \nu \mapsto \nu(B)$ measurable when $B \in \mathcal{D}$).

The *mean measure* $\lambda(\cdot)$ of the point process N is denoted by $\lambda(B) = \mathbf{E}[N(B)]$, $B \in \mathcal{D}$. In this paper we will consider only point processes whose mean measure is locally finite.

Here are two examples of point processes.

Example 2.1. The most prominent point processes are *Poisson processes* defined as follows: given a non-negative, locally finite measure λ on $\mathbb{D}$, N is Poisson on $\mathbb{D}$ with mean measure λ if for each bounded $A \in \mathcal{D}$ the random variable $N(A)$ is Poisson with mean $\lambda(A)$ and for all mutually disjoint $A_1 \subset A_k \in \mathcal{D}$ the random variables $N(A_i), \ldots, N(A_k)$ are independent.

Example 2.2. Another important class of point process are Gibbs processes. For a given nonnegative measurable function $\bar{\mathcal{E}} : \mathbb{M} \to \mathbb{R}_+$ and a measure λ on $\mathbb{D}$, the *Gibbs distribution* on $\mathbb{M}$, with *energy function* $\bar{\mathcal{E}}$ and Poisson *weight process* N of mean measure λ, is the distribution $\Pi_{\bar{\mathcal{E}}}$ on $\mathbb{M}$ defined by

$$\Pi_{\bar{\mathcal{E}}}(\Gamma) = Z^{-1}\mathbf{E}[\mathbb{1}(N \in \Gamma)\bar{\mathcal{E}}(N)]\,, \quad \Gamma \in \mathcal{M}\,,$$

where $Z = \mathbf{E}[\bar{\mathcal{E}}(N)]$ is the normalizing constant called also *partition function* or *statistical sum*, which is assumed to be positive and finite. The energy function can often be expressed as follows

$$-\log(\bar{\mathcal{E}}(\nu)) = \sum_{k=1}^{\nu(\mathbb{D})} \mathcal{E}(x_k, \textstyle\sum_{i=1}^{k-1} \varepsilon_{x_i})\,,$$

where $\nu = \sum_{i=1}^{\nu(\mathbb{D})} \varepsilon_{x_i}$, and where $\mathcal{E} : \mathbb{D} \times \mathbb{M} \mapsto \mathbb{R}$ is called the *local energy function.*

2.2. Measure-valued Markov process

We will model the evolution of the system of particles over time by a time-homogeneous Markov jump process $\{N_t;\, t \geq 0\}$ taking its values in the state space $\mathbb{M}$ of all finite counting measures on $\mathbb{D}$.

Recall that $\mathbb{M}$ is a Polish space and, very much as in (see e.g. [8, Ch. 1]), we will call a family $\{P_t(\nu, \cdot)\}$, $t \geq 0$, $\nu \in \mathbb{M}$ of (possibly defective; i.e, $P_t(\nu, \mathbb{M}) \leq 1$) probability measures on $(\mathbb{M}, \mathcal{M})$ a *(sub-stochastic) Markov kernel* or *transition function* of a jump process if $P_t(\cdot, \Gamma)$ is $\mathcal{D}$-measurable for each $\Gamma \in \mathcal{M}$, $t \geq 0$ and the following two conditions hold:

- *Chapman-Kolmogorov equation:*
$$P_{t+s}(\nu,\Gamma) = \int_{\mathbb{M}} P_s(\mu,\Gamma)\, P_t(\nu,\mathrm{d}\mu) \tag{2.1}$$
for all $t,s\ge 0$, $\nu\in\mathbb{M}$, $\Gamma\in\mathcal{M}$,
- *continuity:* for each $\nu\in\mathbb{M}$, $\Gamma\in\mathcal{M}$,
$$\lim_{t\searrow 0} P_t(\nu,\Gamma) = P_0(\nu,\Gamma) = \mathbb{1}(\nu\in\Gamma)\,,$$
where $\mathbb{1}(A)=1$ if A true and 0 otherwise.

Any process $\{N_t : t\ge 0\}$ satisfying $\mathbf{P}(N_t\in\Gamma \,|\, N_0=\nu) = P_t(\nu,\Gamma)$ is a called *Markov jump process* with kernel $\{P_t(\cdot,\cdot)\}$. Moreover, we will often call the kernel $\{P_t(\cdot,\cdot)\}$ itself a Markov jump process. Note that N_t might not be defined for all $t\ge 0$ when $\{P_t(\cdot,\cdot)\}$ is sub-stochastic.

Given a Markov kernel $\{P_t(\cdot,\cdot)\}$ one defines its *infinitesimal generator* $(q(\cdot),q(\cdot,\cdot))$ (also called *q-pair*)

$$\begin{aligned} q(\nu,\Gamma) &= \lim_{t\searrow 0} t^{-1} P_t(\nu,\Gamma\setminus\{\nu\}), \quad \nu\in\mathbb{M}\,,\Gamma\in\mathcal{M}\,, && (2.2)\\ q(\nu) &= \lim_{t\searrow 0} t^{-1}\big(1-P_t(\nu,\{\nu\})\big), \quad \nu\in\mathbb{M}\,. && (2.3)\end{aligned}$$

Conversely, suppose that some q-pair is given and is *stable*, i.e. that it is such that $q(\nu)<\infty$ for all $\nu\in\mathbb{M}$, and *conservative*, i.e. that $q(\nu)=q(\nu,\mathbb{M}\setminus\{\nu\})$, for all $\nu\in\mathbb{M}$. Then, this q-pair uniquely defines a *minimal* Markov jump process $\{N_t^{(\infty)}\}$. Namely, one defines its kernel $P_t^{(\infty)}$ as the *minimal solution* of the *Backward Kolmogorov equations*:

$$\begin{aligned} \frac{\mathrm{d}P_t(\nu,\Gamma)}{\mathrm{d}t} &= -P_t(\nu,\Gamma)q(\nu) + \int_{\mathbb{M}} P_t(\mu,\Gamma)\, q(\nu,\mathrm{d}\mu)\,, \quad \nu\in\mathbb{M},\Gamma\in\mathcal{M}\,, && (2.4)\\ P_0(\nu,\Gamma) &= \mathbb{1}(\nu\in\Gamma) \quad \nu\in\mathbb{M},\Gamma\in\mathcal{M}\,; \end{aligned}$$

$P_t^{(\infty)}$ is a possibly sub-stochastic Markov kernel describing the evolution of $N_s^{(\infty)}(\omega)$ for $t\in[0,t_\infty(\omega))\}$, where $t_\infty(\omega)$ is the explosion time, that is the first accumulation point of jump times (possibly at ∞). The process $\{N_s^{(\infty)} : s\in[0,t]\}$ has a finite number of jumps for any $t<t_\infty(\omega)$.

If the minimal solution $P_t^{(\infty)}$ is stochastic, i.e., $P_t^{(\infty)}(\nu,\mathbb{M})=1$ for all $\nu\in\mathbb{M}$ and $t\ge 0$, or equivalently $t_\infty=\infty$ almost surely (a.s.), then $P_t^{(\infty)}(\cdot,\cdot)$ is the unique Markov kernel solving (2.4) . In this case we say that q is *regular* and denote this unique solution by $P_t(\cdot,\cdot)=P_t^{(\infty)}(\cdot,\cdot)$ (see e.g. [8, Ch. 2]).

Suppose q given and let $P_t(\cdot,\cdot)$ be the associated Markov kernel. We say that a non-null measure Π on $\mathbb{M}$ is *invariant* for $P_t(\cdot,\cdot)$ if

$$\Pi(\Gamma) = \int_{\mathbb{M}} P_t(\nu,\Gamma)\,\Pi(\mathrm{d}\nu)\,, \quad \Gamma\in\mathcal{M}, t\ge 0\,. \tag{2.5}$$

It is known that (2.5) is equivalent to the following set of *global balance equations*

$$\int_\Gamma q(\nu,\mathbb{M})\Pi(\mathrm{d}\nu) = \int_{\mathbb{M}} q(\mu,\Gamma)\,\Pi(\mathrm{d}\mu)\,, \quad \Gamma\in\mathcal{M}\,; \tag{2.6}$$

(see e.g. [8, Theorem 4.17, p. 129]).

The Markov kernel $P_t(\cdot,\cdot)$ (or the associated process $\{N_t;\, t \geq 0\}$) is called *ergodic* if there exists a probability measure Π satisfying

$$\lim_{t\to\infty} \sup_{\Gamma\in\mathcal{M}} |P_t(\nu,\Gamma) - \Pi(\Gamma)| = 0$$

for all $\nu \in \mathbb{M}$, i.e. when $P_t(\nu,\cdot)$ converges to $\Pi(\cdot)$ *in total variation.*

The Markov kernel $P_t(\cdot,\cdot)$ is called *reversible* with respect to a non-null measure Π on $\mathbb{M}$ if

$$\int_{\Gamma_1} P_t(\nu,\Gamma)\Pi(\mathrm{d}\nu) = \int_{\Gamma} P_t(\nu,\Gamma_1)\,\Pi(\mathrm{d}\nu)\,, \quad \Gamma,\Gamma_1 \in \mathcal{M}, t \geq 0\,. \tag{2.7}$$

It is known that (2.7) is equivalent to the set of *detailed balance equations* which have the form of (2.6) with $\mathbb{M}$ replaced by Γ_1, for all $\Gamma_1 \in \mathcal{M}$. (see e.g. [8, Theorem 6.7 p. 230]). In what follows we will denote by $\mathbf{E}_\nu[\cdot] = \mathbf{E}[\cdot\,|\,N_0 = \nu]$ the conditional expectation of the process $\{N_t\}$ given initial value $N_0 = \nu$. Moreover, we will denote by $\mathbf{E}_\Pi[\cdot] = \int_{\mathbb{M}} \mathbf{E}_\nu[\cdot]\,\Pi(\mathrm{d}\nu)$ the expectation corresponding to the distribution of the Markov process $\{N_t\}$ with initial distribution Π.

3. Spatial Markov queueing process

In this section we introduce the spatial Markov queueing (SMQ) and spatial birth-and-death (SBD) processes. We first give sufficient conditions on the regularity of the SMQ generator and sufficient conditions on its ergodicity. Our approach is inspired by [13], where SBD processes are studied.

We will need a "virtual" state $o \notin \mathbb{D}$ which can be seen as a location outside the space $\mathbb{D}$, and which represents the initial location of particles arriving to or leaving the system. Denote $\overline{\mathbb{D}} = \mathbb{D} \cup \{o\}$ and take $\bar{\mathcal{D}} = \mathcal{D} \cup \{\Gamma \cup \{o\} : \Gamma \in \mathcal{D}\}$ as the σ-field on $\overline{\mathbb{D}}$.

Define the following *displacement operator* T on the space $\mathbb{M}$: for $\nu \in \mathbb{M}, x, y \in \mathbb{D}$

$$\begin{aligned}
T_{oy}\nu &= \nu + \varepsilon_y,\\
T_{xo}\nu &= \nu - \varepsilon_x \quad \text{defined only if } \nu(\{x\}) \geq 1,\\
T_{xy}\nu &= \nu - \varepsilon_x + \varepsilon_y, \quad \text{defined only if } \nu(\{x\}) \geq 1\,.
\end{aligned}$$

The transition $\nu \to T_{ox}\nu$ will be called the *birth* of a particle at x; the transition $\nu \to T_{xo}\nu$ is the *death* of a particle at x, which is well defined provided $\nu(x) > 0$; the transition $\nu \to T_{xy}\nu$, $x, y \in \mathbb{D}$ is the displacement of a particle from x to y.

3.1. Infinitesimal generator

Consider the following infinitesimal generator: for $\nu \in \mathbb{M}$, $\Gamma \in \mathcal{M}$ let

$$\begin{aligned}
q(\nu,\Gamma) &= \int_{\mathbb{D}} r(\nu, T_{oy}\nu)\mathbb{1}(T_{oy}\nu \in \Gamma)\,\lambda(o,\mathrm{d}y) \\
&\quad + \int_{\mathbb{D}} \nu(\mathrm{d}x) \int_{\overline{\mathbb{D}}} r(\nu, T_{xy}\nu)\mathbb{1}(T_{xy}\nu \in \Gamma)\,\lambda(x,\mathrm{d}y)\,, \\
q(\nu) &= q(\nu,\mathbb{M})\,,
\end{aligned} \tag{3.1}$$

where

- $\lambda(x, A)$ ($x \in \overline{\mathbb{D}}$, $A \in \bar{\mathcal{D}}$) is a kernel on the state space $\overline{\mathbb{D}}$ called the *routing kernel*; it describes the dynamics of the births, displacements and deaths of a single hypothetical particle in the absence of any queueing or blocking phenomena. **In what follows we always assume that** $0 \le \lambda(x, \overline{\mathbb{D}}) < \infty$ **and** $\lambda(x, \{x\}) = 0$ **for all** $x \in \overline{\mathbb{D}}$**.**
- $r(\nu, T_{xy}\nu)$ ($x, y \in \overline{\mathbb{D}}$, $\nu \in \mathbb{M}$ and $\nu(\{x\}) \ge 1$ for $x \neq o$) is the *departure-arrival* rate for the displacement form x to y; it describes the rate at which a single particle in the configuration ν is "repulsed" form its location x and is "attracted" by a new location y, with the repulsion and the attraction possibly being dependent of the entire configuration ν. **In what follows we always assume that** $0 \le r(\nu, T_{xy}\nu) < \infty$ **for all** $\nu \in \mathbb{M}$**,** $x, y \in \overline{\mathbb{D}}$ **and** $x \in \nu$ **or** $x = o$**.**

Note that by definition q is conservative and by our assumptions on λ and r it is stable if

$$\int_{\mathbb{D}} r(\nu, T_{xy}\nu)\, \lambda(x, \mathrm{d}y) < \infty$$

for all $\nu \in \mathbb{M}$, $x \in \nu$ and $x = o$, **what will be assumed from now on.**

Remark 3.1. In the queueing literature $q(\cdot, \cdot)$ is called *Whittle network generator* if the departure-arrival rates have the form $r(\nu, T_{xy}\nu) = \psi_x(\nu)$ and *Jackson network generator* if $\psi_x(\nu) = \psi_x(\nu(\{x\}))$ (see e.g. [15]). Note that the open spatial Markov queueing network considered in [15, Chapter 10] is a Whittle network, and thus a special case of our SMQ process. If $r(\nu, T_{xy}\nu) = 0$ when $x, y \in \mathbb{D}$, then $q(\cdot, \cdot)$ is the generator of a *spatial birth-and-death process* (see e.g. [13]).

It is customary to introduce some extra notation. Let

$$T_{AB}\nu = \Big\{T_{xy}\nu : x \in A, y \in B, x \neq y\Big\}, \quad A, B \in \bar{\mathcal{D}}, \nu \in \mathbb{M}\,.$$

With a slight abuse of the above notation we will write also $T_{oB}\nu = T_{\{o\}B}\nu$ and $T_{Ao}\nu = T_{A\{o\}}\nu$. Hence $q(\nu, \Gamma)$ in (3.1) may be written as the sum

$$q(\nu, \Gamma) = q(\nu, \Gamma \cap T_{o\mathbb{D}}\nu) + q(\nu, \Gamma \cap T_{\mathbb{D},\mathbb{D}}\nu) + q(\nu, \Gamma \cap T_{\mathbb{D}o}\nu)\,,$$

where the three terms correspond, respectively, to the intensity of births, displacements and deaths.

Example 3.1. The *Markov-Poisson location (MPL)* process can be seen as a SMQ process where particles are born, move in $\mathbb{D}$ or leave $\mathbb{D}$ independently of each other. Thus, $r(\nu, T_{xy}\nu) \equiv 1$.

Example 3.2. A *spatial brith-and-death processes* (SBD) is a SMQ process without mobility. Thus $\lambda(x, \mathbb{D}) = 0$ for all $x \in \mathbb{D}$.

3.2. Regularity

We aim to establish sufficient conditions for q to be regular. We begin by the result for a particular case that will be used to prove the main result of this section.

Proposition 3.1. *Let q be a generator given by* (3.1)*. If*

$$\sup_{\nu \in \mathbb{M}} q(\nu, T_{o\mathbb{D}}\nu) + q(\nu, T_{\mathbb{D}\mathbb{D}}\nu) < \infty, \tag{3.2}$$

then q is regular.

Proof. Consider the minimal process $\{N_t^{(\infty)} : t \in [0, t_\infty)\}$ associated with q. Recall that q is regular if $t_\infty \equiv \infty$ or, equivalently, if the number of jumps of $\{N_t^{(\infty)}\}$ in any finite time interval is finite with probability 1. Consider the renewal process $S_n = \sum_{k=1}^n Y_k$, where $Y_1, Y_2, \ldots$ are i.i.d. exponential random variables with parameter $\sup_{\nu\in\mathbb{M}} q(\nu, T_{o\mathbb{D}}\nu) + q(\nu, T_{\mathbb{D}\mathbb{D}}\nu)$. Using (3.2) and the strong law of large numbers, we obtain that $\lim_{n\to\infty} S_n = \infty$ a.s. We can construct our minimal Markov process together with the renewal process $\{S_n\}$ on a common probability space assuring that the (increasing) sequence of birth and motion instants, denoted by $0 \le T_1 < T_2, \ldots$, satisfies $T_n \ge S_n$ a.s. By this construction, the number of births and motions in each finite time interval is less than the number of renewals of $\{S_n\}$ and it is thus finite. The number of deaths of the Markov process is at most equal to the number of births and thus it is also finite. This completes the proof. □

Define

$$b_n = \sup_{\nu\in\mathbb{M}:\nu(\mathbb{D})=n} q(\nu, T_{o\mathbb{D}}\nu)\,, \tag{3.3}$$

$$d_n = \inf_{\nu\in\mathbb{M}:\nu(\mathbb{D})=n} q(\nu, T_{\mathbb{D}o}\nu)\,. \tag{3.4}$$

From now on we will assume that for each $n \ge 0$

$$b_n < \infty \quad \textbf{and} \quad \sup_{\nu\in\mathbb{M}:\nu(\mathbb{D})=n} q(\nu, T_{\mathbb{D}\mathbb{D}}\nu) < \infty\,. \tag{3.5}$$

The following result gives further sufficient conditions for the generator q to be regular. These conditions are expressed in terms of a classical discrete birth-and-death generator q' with the intensity of births and deaths respectively denoted by b_n and d_n (see Appendix A.1 for more details).

Proposition 3.2. *Let q be a generator given by* (3.1)*. If the discrete birth-and-death generator q', with the intensity of births and deaths b_n, d_n respectively defined by* (3.3) *and* (3.4)*, is regular then q is regular.*

Before the proof we make some remarks.

Remark 3.2. Proposition 3.2 can be seen as en extension of results on the regularity of SBD processes in [13]. Note that Proposition 3.2 combined with the results on the regularity of discrete birth-and-death generators (see Appendix A.1) gives weaker conditions on regularity of a discrete birth-mobility-and-death process than these of [8, Theorem 3.19], where $\sup_{\nu:\nu(\mathbb{D})\le n} q(\nu) < \infty$ is assumed for all $n \ge 0$; which implies in particular that the death rates are uniformly bounded over $\{\nu : \nu(\mathbb{D}) \le n\}$. Note also that [15, Chapter 10] ignores regularity conditions of SMQ.

We will give a proof of Proposition 3.2 based on a coupling argument analogous to that used in [13]. In the sequel we define the generator of some Markov process that couples the spatial birth-mobility-and-death process with a discrete birth-and-death process in such a way that at any time the total number of particles of the spatial process is at most equal to the value of the discrete process. This coupling will also be used when we will study ergodicity.

Consider an infinitesimal generator $\tilde{q}$ on $\tilde{\mathbb{M}} = \mathbb{M} \times \mathbb{N}$ defined as follows.

- If $\nu(\mathbb{D}) \neq n$, then

$$\begin{cases} \tilde{q}\left((\nu,n),\Gamma\times\{n\}\right) & = q(\nu,\Gamma)\ , \quad \Gamma\subset\mathbb{M} \\ \tilde{q}\left((\nu,n),\{\nu\}\times\{n+1\}\right) & = b_n \\ \tilde{q}\left((\nu,n),\{\nu\}\times\{n-1\}\right) & = d_n \end{cases}$$

$$\tilde{q}(\nu,n) = q(\nu) + b_n + d_n$$

- If $\nu(\mathbb{D}) = n$, then

$$\begin{cases} \tilde{q}\left((\nu,n),\Gamma\times\{n+1\}\right) & = q(\nu,\Gamma)\,, \quad \Gamma\subset T_{o\mathbb{D}}\nu \\ \tilde{q}\left((\nu,n),\{\nu\}\times\{n+1\}\right) & = b_n - q(\nu,T_{o\mathbb{D}}\nu) \\ \tilde{q}\left((\nu,n),\Gamma\times\{n-1\}\right) & = q(\nu,\Gamma)\,\frac{d_n}{q(\nu,T_{\mathbb{D}o}\nu)}\,, \quad \Gamma\subset T_{\mathbb{D}o}\nu \\ \tilde{q}\left((\nu,n),\Gamma\times\{n\}\right) & = q(\nu,\Gamma)\left(1-\frac{d_n}{q(\nu,T_{\mathbb{D}o}\nu)}\right)\,, \quad \Gamma\subset T_{\mathbb{D}o}\nu \\ \tilde{q}\left((\nu,n),\Gamma\times\{n\}\right) & = q(\nu,\Gamma)\,, \quad \Gamma\subset T_{\mathbb{D}\mathbb{D}}\nu \end{cases}$$

$$\tilde{q}(\nu,n) = b_n + q(\nu,T_{\mathbb{D}o}\nu) + q(\nu,T_{\mathbb{D}\mathbb{D}}\nu)\,.$$

Note that for all $(\nu,n)\in\tilde{\mathbb{M}}$ and all $\Gamma\neq\{\nu\}$, $\tilde{q}\left((\nu,n),\Gamma\times\mathbb{N}\right) = q(\nu,\Gamma)$, and that in addition $\tilde{q}\left((\nu,n),\mathbb{M}\times\{n+1\}\right) = b_n$, and $\tilde{q}\left((\nu,n),\mathbb{M}\times\{n-1\}\right) = d_n$, which means that $\tilde{q}$ is a *coupling* of the two generators q and q' mentioned in Proposition 3.2. Consider the minimal Markov process $\{\tilde{N}_t\} = \{(N_t,N'_t)\}$ associated to $\tilde{q}$ with an initial value $\tilde{N}_0 = (N_0,N'_0)$. Then for $t<\tilde{t}_\infty$, where $\tilde{t}_\infty$ is the explosion time of $\tilde{N}_t$, $\{N_t\}$ is a Markov process on $\mathbb{M}$ with generator q and initial condition N_0 whereas $\{N'_t\}$ is a Markov process on the integers with generator q' and initial condition N'_0. Moreover it is easy to check that by construction, $\{\tilde{N}_t\}$ has the following important property:

Remark 3.3. If the initial states are such that $N_0(\mathbb{D})\leq N'_0$, then $N_t(\mathbb{D})\leq N'_t$ for all $0\leq t<\tilde{t}_\infty$.

For all integers m and n, let

$$\begin{aligned} \mathbb{M}_m &= \{\nu\in\mathbb{M}:\nu(\mathbb{D})\leq m\} \\ \tilde{\mathbb{M}}_{m,n} &= \mathbb{M}_m\times\{0,1,\ldots,n\}. \end{aligned}$$

We will use the following lemma to prove Proposition 3.2.

Lemma 3.1. *Consider the minimal Markov process* $\{\tilde{N}_t\} = \{(N_t,N'_t)\,,t\in[0,\tilde{t}_\infty)\}$ *associated to* $\tilde{q}$*. Then for all* $m\in\mathbb{N}$

$$\mathbf{P}\{\,\tilde{N}_s\in\tilde{\mathbb{M}}_{m,m} \text{ for all } s\in[0,\tilde{t}_\infty) \text{ and } \tilde{t}_\infty<\infty\,\} = 0\,.$$

Proof. For all $t\geq 0$, let $X(t)$ denote the number of jumps of $\{\tilde{N}_s\}$ in $[0,t]$. Let T denote the time of the first transition of $\tilde{N}_s$ to a state outside $\tilde{\mathbb{M}}_{m,m}$ ($\tilde{t}_\infty$ if there is no such transition). Note that $X(T\wedge t)$ may be viewed as the number of jumps in $(0,t]$ of the minimal Markov process with the generator $\tilde{q}^{m,m}$ where each state outside $\tilde{\mathbb{M}}_{m,m}$ is made absorbing. By Assumption (3.5) and Proposition 3.1 this generator is regular and thus $X(T\wedge t)$ is a.s. finite.

In order to prove the lemma, is enough to show that

$$Y(t) = X(t) \times \mathbb{1}\left(\tilde{N}_s \in \tilde{\mathbb{M}}_{m,m},\ \text{for all } s \in \left[0, t \wedge \tilde{t}_\infty\right)\right)$$

is a.s. finite for all t. But since we have $Y(t) \le X(T \wedge t)$, $Y(t)$ is finite indeed. □

Proof. (Proposition 3.2) Let q, q' be as in Proposition 3.2 and let $\tilde{q}$ be the coupling defined above. For $\Gamma \subset \mathbb{M}$ and $\nu \in \mathbb{M}$, we denote by $\mathcal{P}_t^m(\nu, \Gamma)$ the following taboo probability: this is the probability for the minimal Markov process with generator q, to go from ν at time 0 to Γ at time t (which implies $t_\infty > t$) and to avoid $\mathbb{M}_m^c$ (or equivalently to remain inside the set $\mathbb{M}_m$) on the whole time interval $[0,t]$. Analogously, we define $\mathcal{P}_t'^m(n, A)$ (resp. $\tilde{\mathcal{P}}_t^{k,m}((\nu, n), \Gamma \times A)$) to be the taboo probability for the minimal process associated with q' (resp. $\tilde{q}$), to go from n (resp. (ν, n)) at time 0 to A (resp. $\Gamma \times A$) at time t and to remain in $\{0, 1, \ldots, m\}$ (resp. $\tilde{\mathbb{M}}_{k,m}$) on the whole time interval $[0,t]$.

From Remark 3.3, for all integers m, for all (ν, n) such that $\nu(\mathbb{D}) \le n \le m$ and for all $\Gamma \subset \tilde{\mathbb{M}}$

$$\tilde{\mathcal{P}}_t^{k,m}((\nu, n), \Gamma \times A) = \tilde{\mathcal{P}}_t^{m,m}((\nu, n), \Gamma \times A)\,, \qquad \forall k \ge m. \tag{3.6}$$

We now prove that for all $t \ge 0$, and all (ν, n) and $\Gamma \subset \tilde{\mathbb{M}}$ as above,

$$\tilde{\mathcal{P}}_t^{m,m}((\nu, n), \mathbb{M} \times A) = \mathcal{P}_t'^m(n, A)\,. \tag{3.7}$$

Obviously we have the inequality $\tilde{\mathcal{P}}_t^{m,m}((\nu, n), \mathbb{M} \times A) \le \mathcal{P}_t'^m(n, A)$. To prove the equality, by (3.6), it suffices to know that the probability that $t > \tilde{t}_\infty$ and $\tilde{N}_s \in \tilde{\mathbb{M}}_{m,m}$ for all $s \in [0, \tilde{t}_\infty)$ is 0. This hold true by Lemma 3.1, which concludes the proof of (3.7).

By the monotone continuity property of probabilities,

$$\lim_{m \to \infty} \tilde{\mathcal{P}}_t^{m,m}((\nu, n), \mathbb{M} \times A) = \tilde{P}_t^{(\infty)}((\nu, n), \mathbb{M} \times A), \tag{3.8}$$

$$\lim_{m \to \infty} \mathcal{P}_t'^m(n, A) = P_t'^{(\infty)}(n, A)\,, \tag{3.9}$$

where $\tilde{P}_t^{(\infty)}$, $P_t'^{(\infty)}$ respectively denote the minimal solutions of the backward Kolmogorov equations with generators $\tilde{q}$ and q'. Then, by (3.7)

$$\tilde{P}_t^{(\infty)}((\nu, n), \mathbb{M} \times A) = P_t'^{(\infty)}(n, A)$$

and in particular

$$\tilde{P}_t^{(\infty)}((\nu, n), \mathbb{M} \times \mathbb{N}) = P_t'^{(\infty)}(n, \mathbb{N}) = 1,$$

provided q' is regular. To conclude the proof observe that

$$\tilde{P}_t^{(\infty)}((\nu, n), \mathbb{M} \times \mathbb{N}) \le P_t^{(\infty)}(\nu, \mathbb{M}).$$

□

3.3. Ergodicity

A state x_0 of a Markov jump process $\{X_t\}$ is said to be *positive recurrent* if $\mathbf{E}[T|X_0 = x_0] < \infty$, where T is the return time of X_t to x_0 after the first jump of the process.

In this section we suppose that the conditions of Proposition 3.2 are satisfied and we give sufficient conditions for q to be ergodic. We again use the coupling generator $\tilde{q}$ introduced in the previous section.

Since q is regular we will denote the unique Markov kernel associated to it by $P_t(\cdot, \cdot)$. Let $\{N_t\}$ be the right-continuous-path jump Markov process driven by $P_t(\cdot, \cdot)$ and some distribution for N_0. Let $\emptyset \in \mathbb{M}$ denote the null measure: $\emptyset(A) \equiv 0$ for any $A \in \mathcal{D}$.

Proposition 3.3. *Suppose that the state* 0 *is positive recurrent for the discrete birth-and-death generator* q' *with parameters* b_n *and* d_n *respectively defined by* (3.3) *and* (3.4)*. Then* $\emptyset$ *is a positive recurrent state for the generator* q *and the limit*

$$\Pi(\Gamma) = \lim_{t \to \infty} P_t\left(\emptyset, \Gamma\right) \tag{3.10}$$

exists for all $\Gamma \in \mathcal{M}$ *and it is given by*

$$\Pi\left(\Gamma\right) = \frac{1}{\mathbf{E}_\emptyset\left[T\right]} \mathbf{E}_\emptyset \left[\int_0^T \mathbb{1}\left(N_t \in \Gamma\right) \mathrm{d}t\right], \tag{3.11}$$

where T *is the return time of* N_t *to state* $\emptyset$ *and* $\mathbf{E}_\emptyset\left[\cdot\right]$ *is the conditional expectation* $\mathbf{E}_\emptyset\left[\cdot\right] = \mathbf{E}\left[\cdot | N_0 = \emptyset\right]$.

Proof. Let T' denote the return time to 0 of the process N' associated to q'. By assumption, $\mathbf{E}_0[T'] < \infty$. In order to show that $\mathbf{E}_\emptyset[T] < \infty$, we use the coupling process $\tilde{N}_t = (N_t, N'_t)$, and the relation

$$\mathbf{E}_\emptyset[T] = \mathbf{E}_{(\emptyset,0)}[T] \leq \mathbf{E}_{(\emptyset,0)}[T'] = \mathbf{E}_0[T'] < \infty\,,$$

where the inequality follows form Remark 3.3 and the fact that the first jump of the coupling process from the state $(\emptyset, 0)$ is a joint birth on both coordinates N_t and N'_t (indeed, $b_0 = q(\emptyset, T_{o\mathbb{D}}\emptyset)$ and thus $\tilde{q}((\emptyset, 0), \{\emptyset\} \times \{1\}) = 0$).

Now the weak convergence property (3.10) and the limiting distribution (3.11) follow from the standard arguments for regenerative processes (see e.g. [1, Theorem 1.2 p.170]). This theorem can be used here since $\mathbb{M}$ is a metrizable space and since the distribution of the regeneration time T is non-lattice (in fact it is absolutely continuous with respect to Lebesgue's measure). The convergence for all $\Gamma \in \mathcal{M}$ can be shown following the proof of Theorem 1.2 in [1] and observing that the function $\mathbf{E}_\emptyset[\mathbb{1}(N_t \in \Gamma); t < T]$ is right continuous in t.

□

Note that the question of ergodicity of $\{N_t\}$ is not studied in Proposition 3.3. In particular, the question of irreducibility was not addressed. Within this setting, nothing excludes that there exist other limiting measures than (3.11). Similarly, the convergence result (3.10) has no reason to hold when the initial condition is not $\emptyset$. The following result tackles this problem.

Let T' denote the return time of the discrete birth-and-death process N' with generator q' to state 0.

Corollary 3.1. *If the conditions of Proposition* 3.3 *are satisfied and if for some* $n \in \mathbb{N}$, $\mathbf{P}\left\{ T' < \infty | N_0' = n \right\} = 1$, *then*

$$\lim_{t\to\infty} P_t\left(\nu, \Gamma\right) = \Pi\left(\Gamma\right),$$

for all $\nu \in \mathbb{M}$ *such that* $\nu(\mathbb{D}) \le n$, *where* Π *is given by* (3.11).

Proof. In view of [1, Theorem 1.2 p.170] is suffices to show that the return time of N_t from any $\nu \in \mathbb{M}$, $\nu(\mathbb{D}) \le n$, to $\emptyset$ is almost surely finite. This is a consequence of Remark 3.3 and the assumption on T'.

□

The following result strengthens the convergence stated in Proposition 3.3 and Corollary 3.1.

Corollary 3.2. *The convergence in Corollary* 3.1 *holds in the sense of total variation.*

Proof. The result follows from [1, Corollary 1.4 p. 188]. Because $\{N_t\}$ is a pure jump Markov process so $N_t(\omega)$ is measurable jointly in (t,ω); in addition, the regeneration time has a spread-out distribution (in fact absolutely continuous with respect to Lebesgue's measure).

Remark 3.4. The results of this section, in particular of Corollary 3.2, can be used to establish sufficient conditions on ergodicity of SMQ in the sense of Section 2.2. Thy extend also the results on ergodicity of SBD in [13]. Here are a few comments on the ergodicity result for spatial queueing systems given in [15, Chapter 10]. Recall (see Remark 3.1) that the open spatial queueing system considered there is a special case of our SMQ process, where $r(\nu, T_{xy}\nu) = \psi_x(\nu)$ for some function $\psi_{\cdot}(\cdot)$. In this case [15, Theorem 10.5] considers (instead of our dominating discrete birth-and-death process) a MPL process with routing kernel

$$\hat{\lambda}(x,B) = \bar{b}_x \lambda(x,B) \qquad x \in \overline{\mathbb{D}}, B \subset \overline{\mathcal{D}},$$

where

$$\bar{b}_x = \begin{cases} \inf_{\nu \ne \emptyset} \psi_x(\nu) & \text{for } x \in \mathbb{D} \\ \sup_\nu \psi_o(\nu) & \text{for } x = o\,, \end{cases}$$

provided $\bar{b}_0 < \infty$ and $\bar{b}_x > 0$ for $x \in \mathbb{D}$. As observed in the proof of [15, Theorem 10.5], it is possible to couple the original Whittle process $\{N_t\}$ with this MPL process, say $\{\hat{X}_t\}$, in such a way that $N_t(\mathbb{D}) \le \hat{X}_t(\mathbb{D})$ for all $t \ge 0$. Then, as observed in this proof too, for $\emptyset$ to be a positive recurrent state of $\{N_t\}$, it suffices to assume that the same holds true for $\{\hat{X}_t\}$. The necessary and sufficient condition for the latter is

$$\int_{\mathbb{D}} 1/\bar{b}_x\, \rho(\mathrm{d}x) < \infty\,, \tag{3.12}$$

where $\rho(\cdot)$ is the solution of the traffic equations (see Section 3.4), which is a much weaker condition than $\bar{b}_o \int_{\mathbb{D}} 1/\bar{b}_x\, \rho(\mathrm{d}x) < 1$ used in [15] (c.f. condition (10.11) there). Indeed, the time between successive visits of $\{\hat{X}_t\}$ to $\emptyset$ is the duration of the busy period in an $M/G/\infty$ queue, and thus its expectation is finite under the (necessary and sufficient) condition that the service time in this queue has a finite expectation, which is equivalent to (3.12).

Note also that in the case of a Whittle network, our dominating birth-and-death process has the following rates

$$b_n = \sup_{\nu(\mathbb{D})=n} \lambda(o,\mathbb{D})\psi_o(\nu) \le \bar{b}_o\lambda(o,\mathbb{D})\,, \tag{3.13}$$

$$d_n = \inf_{\nu(\mathbb{D})=n} \int_{\mathbb{D}} \psi_x(\nu)\lambda(x,\{o\})\,\nu(\mathrm{d}x) \ge \inf_{\nu(\mathbb{D})=n} \int_{\mathbb{D}} \bar{b}_x\lambda(x,\{o\})\,\nu(\mathrm{d}x)\,. \tag{3.14}$$

In contrast to [15, Theorem 10.5], we do not require $\bar{b}_o < \infty$, which would imply $\sup_n b_n < \infty$. Our Proposition 3.3 combined with the results on the ergodicity of the discrete birth-and-death process (see Appendix A.1) gives a way to handle the case where the sequence b_n is unbounded. Indeed suppose that

$$\inf_{x\in\mathbb{D}} \bar{b}_x\lambda(x,\{o\}) = \epsilon > 0. \tag{3.15}$$

Then we have by (3.14) that $d_n \ge n\epsilon > 0$ and thus by Proposition 3.3 and (A.5) $\emptyset$ is an ergodic state for N_t if

$$\sum_{n=1}^{\infty} \frac{b_0 \dots b_{n-1}}{\epsilon^n n!} < \infty\,. \tag{3.16}$$

For this to hold, the condition $\sup_n b_n < \infty$ is sufficient but not necessary.

As a final comment, note also that our approach to ergodicity does not require an explicit form of the invariant measure and even the existence of the solution to the traffic equations.

3.4. Gibbs invariant measure

In this section we gather results concerning invariant measures for the SMQ process. We begin by a standard observation.

Proposition 3.4. *Under the assumptions of Proposition* 3.3, Π *defined in* (3.11) *is an invariant probability measure; i.e., it satisfies* (2.5) *or equivalently* (2.6)*. Moreover, if Q is an invariant probability measure such that, for Q-almost all $\nu \in \mathbb{M}$, the return time from ν to $\emptyset$ is a finite random variable, then $Q \equiv \Pi$.*

Proof. Invariance of Π*:* Formula (3.11) and the strong Markov property with respect to the natural filtration imply that for Π-almost all $\nu \in \mathbb{M}$, the return time from ν to $\emptyset$ is a finite random variable. Thus for fixed $t \ge 0$ by [1, Theorem 1.2 p.170] (see also the proof of Corollary 3.1) $\lim_{s\to\infty} \Pi P_{s+t}(\Gamma) = \int_{\mathbb{M}} P_{t+s}(\nu,\Gamma)\,\Pi(\mathrm{d}\nu) = \Pi(\Gamma)$. On the other hand

$$\lim_{s\to\infty} \Pi P_{s+t}\,(\Gamma) = \lim_{s\to\infty} \int_{\mathbb{M}} P_t(\nu,\Gamma)\,(\Pi P_s)(\mathrm{d}\nu) = \int_{\mathbb{M}} P_t(\nu,\Gamma)\,\Pi(\mathrm{d}\nu)\,,$$

where the last equality follows from the fact that $P_t(\nu,\Gamma)$ is a bounded function of ν and $\Pi P_s(\Gamma)$ converges for all $\Gamma \in \mathcal{M}$.

Uniqueness: Let Q be an invariant measure i.e. $QP_t(\Gamma) = \int_{\mathbb{M}} P_t(\nu,\Gamma)\,Q(\mathrm{d}\nu) = Q(\Gamma)$ and so $\lim_{t\to\infty} QP_t(\Gamma) = Q(\Gamma)$. The assumption that for Q-almost all $\nu \in \mathbb{M}$, the return time form ν to $\emptyset$ is a finite random variable, and [1, Theorem 1.2 p.170] imply that $\lim_{t\to\infty} QP_t(\Gamma) = \Pi(\Gamma)$. $\square$

Formula (3.11) does not give Π in explicit form. The global balance equation (2.6) can sometimes be used to express Π in a more tractable way. This is the case when the routing kernel $\lambda(\cdot,\cdot)$ satisfies certain traffic equations and $r(\cdot,\cdot)$ is "balanced" in some way that we define in what follows.

We call a locally finite measure $\rho(\cdot)$ on $\overline{\mathbb{D}}$ a *solution of the traffic equations* if

$$\rho(\{o\}) = 1, \text{ and } \quad \int_B \lambda\left(x, \overline{\mathbb{D}}\right) \rho\left(\mathrm{d}x\right) = \int_{\overline{\mathbb{D}}} \lambda\left(y, B\right) \rho\left(\mathrm{d}y\right), \quad \forall B \in \overline{\mathcal{D}}. \tag{3.17}$$

Moreover, we will say that $\lambda(\cdot,\cdot)$ is *reversible with respect to* ρ if (3.17) holds with $\overline{\mathbb{D}}$ replaced by any $A \in \overline{\mathcal{D}}$.

Let $\Psi(\cdot) > 0$ be a measurable function on $\mathbb{M}$. We say that $r(\cdot,\cdot)$ is Ψ*-balanced* if

$$\Psi(\nu) r(\nu, T_{xy}\nu) = \Psi(T_{xy}\nu) r(T_{xy}\nu, \nu), \quad \text{for all } \nu \in \mathbb{M}, x, y \in \overline{\mathbb{D}}, \nu(\{x\}) > 0\,. \tag{3.18}$$

The following result says when a Gibbs distribution is an invariant measure of the SMQ process.

Proposition 3.5. *Consider a SMQ generator q that is regular. Suppose that $\lambda(\cdot,\cdot)$ is reversible with respect to some locally finite measure $\rho(\cdot)$. Suppose that $r(\cdot,\cdot)$ is Ψ-balanced for some positive function $\Psi(\cdot)$. If*

$$\lambda(\overline{\mathbb{D}}) \quad < \quad \infty \tag{3.19}$$

and

$$\int_{\mathbb{M}} \Psi(\nu)\, \Pi_\rho(\mathrm{d}\nu) \quad < \quad \infty\,, \tag{3.20}$$

where Π_ρ is the distribution of the Poisson point process on $\mathbb{D}$ with intensity measure ρ, then Π_Ψ is the Gibbs distribution with energy function Ψ based on the Poisson weight process with intensity ρ is an invariant measure for q. Moreover, q (equivalently $P_t(\cdot,\cdot)$) is reversible with respect to this probability measure.

Remark 3.5. In the case of a Whittle SMQ network, i.e.; when $r(\nu, T_{xy}\nu) = \psi_x(\nu)$, it suffices to assume that ρ is a solution of the balance equations for $\lambda(\cdot,\cdot)$ (and not necessarily that $\lambda(\cdot,\cdot)$ is reversible with respect to ρ) to prove (under all other conditions unchanged) that the Gibbs measure is an invariant measure. The reversibility of q with respect to this Gibbs measure still requires the reversibility of λ. The result of Proposition 3.5 in its full generality is given without proof in [16].

Proof. (*Proposition* 3.5*)* Note that the reversibility of q (or equivalently of $P_t(\cdot,\cdot)$) with respect to Π_Ψ implies the invariance of Π_Ψ. Thus it suffices to prove that

$$I(\Gamma_1, \Gamma_2) = \int_{\Gamma_1} q(\nu, \Gamma_2)\, \Pi_\Psi(\mathrm{d}\nu)$$

is symmetric with respect to Γ_1, Γ_2, i.e. that $I(\Gamma_1, \Gamma_2) = I(\Gamma_2, \Gamma_1)$ for all $\Gamma_1, \Gamma_2 \in \mathcal{M}$. Using the form of q,

$$\begin{aligned} I(\Gamma_1, \Gamma_2) = \int_{\Gamma_1} \biggl(\int_{\mathbb{D}} & r(\nu, T_{oy}\nu) \mathbb{1}(T_{oy}\nu \in \Gamma_2)\, \lambda(o, \mathrm{d}y) \\ & + \int_{\mathbb{D}} \nu(\mathrm{d}x) \int_{\overline{\mathbb{D}}} r(\nu, T_{xy}\nu) \mathbb{1}(T_{xy}\nu \in \Gamma_2)\, \lambda(x, \mathrm{d}y) \biggr) \Pi_\Psi(\mathrm{d}\nu)\,. \end{aligned} \tag{3.21}$$

Note that the second term can be written as

$$\int_{\mathbb{M}}\int_{\mathbb{D}} g(x,\nu-\varepsilon_x)\,\nu(\mathrm{d}x)\Pi_\Psi(\mathrm{d}\nu)\,,$$

where

$$g(x,\nu)=\mathbb{1}(\nu+\varepsilon_x\in\Gamma_1)\int_{\overline{\mathbb{D}}} r(\nu+\varepsilon,T_{oy}\nu)\mathbb{1}(T_{oy}\nu\in\Gamma_2)\,\lambda(x,\mathrm{d}y)\,,$$

interpreting $T_{oo}\nu\equiv\nu$. Since Π_Ψ is the Gibbs distribution defined above, by [17, Theorem 5.1, p. 179], the second term in (3.21) can be written as

$$\begin{aligned}&\int_{\mathbb{M}}\int_{\mathbb{D}} g(x,\nu)\frac{\Psi(\nu+\varepsilon_x)}{\Psi(\nu)}\,\rho(\mathrm{d}x)\Pi_\Psi(\mathrm{d}\nu)\\ &=\int_{\mathbb{M}}\int_{\mathbb{D}}\mathbb{1}(\nu+\varepsilon_x\in\Gamma_1)\int_{\overline{\mathbb{D}}} r(\nu+\varepsilon_x,T_{oy}\nu)\mathbb{1}(T_{oy}\nu\in\Gamma_2)\,\lambda(x,\mathrm{d}y)\frac{\Psi(\nu+\varepsilon_x)}{\Psi(\nu)}\,\rho(\mathrm{d}x)\Pi_\Psi(\mathrm{d}\nu)\,.\end{aligned}$$

Note also that $\rho(\{o\})=1$ and interpreting $\nu+\varepsilon_o\equiv\nu$ we can write the first term in (3.21) as

$$\int_{\mathbb{M}}\mathbb{1}(\nu+\varepsilon_o\in\Gamma_1)\int_{\mathbb{D}} r(\nu+\varepsilon_o,T_{oy}\nu)\mathbb{1}(T_{oy}\nu\in\Gamma_2)\,\lambda(o,\mathrm{d}y)\frac{\Psi(\nu+\varepsilon_o)}{\Psi(\nu)}\rho(\{o\})\Pi_\Psi(\mathrm{d}\nu)$$

Consequently

$$\begin{aligned}&I(\Gamma_1,\Gamma_2)=\\ &\int_{\mathbb{M}}\int_{\overline{\mathbb{D}}}\int_{\overline{\mathbb{D}}}\mathbb{1}(T_{ox}\nu\in\Gamma_1,T_{oy}\nu\in\Gamma_2)r(T_{ox}\nu,T_{oy}\nu)\frac{\Psi(T_{ox}\nu)}{\Psi(\nu)}\;\lambda(x,\mathrm{d}y)\rho(\mathrm{d}x)\Pi_\Psi(\mathrm{d}\nu)\,.\end{aligned}$$

The symmetry of $I(\cdot,\cdot)$ now follows from the reversibility of $\lambda(\cdot,\cdot)$ with respect to $\rho(\cdot)$ and the balance assumption on $r(.,.)$.

□

4. Two spatial loss models

Classical loss models are well adapted to wired communication networks. In wireless communication models, we have to take into account two important aspects, absent in the classical models. The *spatial geometry of the network* can no longer be reduced to an abstract graph of links but has to capture the relative locations of radio channels, which determine their joint feasibility. This spatial component of the model is subject to *changes due to the mobility of users* and also to instantaneous changes of radio conditions. One of the consequences of the above framework is that a call can be rejected not only when it is arriving to the network but also when a mobile user changes its geographical location while his communication is in progress. Note that the latter can happen even if the mobile displacement is the only change in the configuration of calls in progress.

Consider a SMQ generator q introduced in Section 3. We suppose that it is regular and ergodic. We call the corresponding SMQ process $\{N_t\}$ the *free process* and consider it as describing the evolution of a system without capacity constraints. Thus, q describes arrivals of calls, service demands, service discipline and the mobility of calls.

Suppose now that the evolution of the free process is subject to some constraints, which can be expressed as the limitation of the original state space $\mathbb{M}$ to a given fixed measurable subset $\mathbb{M}^f \subset \mathbb{M}$ of feasible states. We will give examples of $\mathbb{M}^f$ used in the modeling of two communication techniques in Section 5. The constrained process, started off at an initial state in $\mathbb{M}^f$ follows the same dynamic as the free process as long as it stays in $\mathbb{M}^f$, and will be forced to modify its behaviour each time an attempt of a transition from $\mathbb{M}^f$ to $\mathbb{M} \setminus \mathbb{M}^f$ occurs. We will consider two possible behaviours adopted at such epochs. They lead to two following different models.

- *Transition blocking model.* In this model we suppose that all the transitions from a state $\nu \in \mathbb{M}^f$ to a state $\mathbb{M} \setminus \mathbb{M}^f$ are "blocked", which means that the process remains in the state ν and continues its evolution driven by q. The dynamics of the constrained process $\{N_t^{tb}\}$ in this model is described by a generator q^{tb} where

$$q^{tb}(\nu,\Gamma) = \begin{cases} q(\nu,\Gamma\cap\mathbb{M}^f) & \text{if } \nu\in\mathbb{M}^f \\ q(\nu,\Gamma) & \text{if } \nu\notin\mathbb{M}^f \end{cases} \tag{4.1}$$

$$q^{tb}(\nu) = \begin{cases} q(\nu,\mathbb{M}^f) & \text{if } \nu\in\mathbb{M}^f \\ q(\nu,\mathbb{M}) & \text{if } \nu\notin\mathbb{M}^f\,. \end{cases}$$

  Note that q^{tb} is also a SMQ generator, with the same routing kernel λ and the departure-arrival rates $r^{tb}(\nu, T_{xy}\nu) = r(\nu, T_{xy}\nu)\mathbb{1}(T_{xy}\nu \in \mathbb{M}^f)$ if $\nu \in \mathbb{M}^f$ and $r(\nu, T_{xy}\nu)$ otherwise; in what follows, we shall always assume that it is regular and ergodic. Moreover q^{tb}, which is called the *truncation* of q, admits an invariant measure which is equal to the truncation of the invariant measure of q to $\mathbb{M}^f$, at least if q^{tb} is reversible. This yields a relatively simple formula for the blocking probability, which we call the spatial Erlang formula.
  Not that in the transition blocking model, there are no losses of calls in progress: an unauthorized displacement is blocked and the call in question remains at its previous location until the next event. This might be seen as a not very realistic assumption in the context of the modeling of voice calls. We will discuss the pertinence of this model in Section 5.
- *Forced termination model.* In this model we suppose that all the call arrivals that would result in taking the process to a state outside $\mathbb{M}^f$ are blocked, as in the transition blocking model; however an attempt of displacement of a call in progress that would take the process to $\mathbb{M}\setminus\mathbb{M}^f$ leads to the forced termination of this call. The evolution of the process is thus described by the following generator

$$q^{ft}(\nu,\Gamma) = \begin{cases} q(\nu,\Gamma\cap\mathbb{M}^f) & \text{for } \Gamma\subset T_{\overline{\mathbb{D}}\mathbb{D}}\nu, \nu\in\mathbb{M}^f\,, \\ q(\nu,\Gamma)+q(\nu,T_{A\mathbb{D}}\nu\setminus\mathbb{M}^f) & \text{for } \Gamma=T_{Ao}\nu, A\subset\mathbb{D}, \nu\in\mathbb{M}^f\,, \\ q(\nu,\Gamma) & \text{for } \Gamma\in\mathcal{M}, \nu\notin\mathbb{M}^f\,, \end{cases} \tag{4.2}$$

$$q^{ft}(\nu) = \begin{cases} q(\nu,\mathbb{M}^f\cup T_{\mathbb{D}\overline{\mathbb{D}}}\nu) & \text{if } \nu\in\mathbb{M}^f\,, \\ q(\nu,\mathbb{M}) & \text{if } \nu\notin\mathbb{M}^f\,, \end{cases}$$

  where we implicitly assume that $\mathbb{M}^f$ is *closed* with respect to transition $T_{xo}\nu$ for all $x \in \mathbb{D}$. Note that q^{ft} is also a SMQ generator, with the same routing kernel

λ and the departure-arrival rates $r^{ft}(\nu, T_{xy}\nu) = r^{tb}(\nu, T_{xy}\nu)$ for $y \neq o$,

$$r^{ft}(\nu, T_{xo}\nu) = r(\nu, T_{xo}\nu) + \int_{\mathbb{D}} r(\nu, T_{x,y}\nu)\mathbb{1}(T_{xy}\nu \notin \mathbb{M}^f)\,\lambda(x, \mathrm{d}y)/\lambda(x, o)$$

if $\nu \in \mathbb{M}^f$ and $r(\nu, T_{xo}\nu)$ otherwise. We will always assume that q^{ft} is regular and ergodic. However it cannot be seen as a truncation of q and typically its invariant measure is not explicitly known even if the invariant measure of q is known.

In the remaining part of this section we will study loss probabilities in the above models.

4.1. Feasibility and blocking probabilities; a spatial Erlang formula

In this section we will concentrate on the transition blocking model. Suppose q is a regular, ergodic SMQ generator, as described in Section 3, and call its unique invariant probability measure Π. Consider the SMQ process $\{N_t\}$ corresponding to q as the free process (without capacity constraints; see the discussion above). Fix a measurable subset $\mathbb{M}^f$ of its state space $\mathbb{M}$ as the subspace of all feasible states of the constrained process $\{N_t^{tb}\}$ that evolves according the generator q^{tb} given by (4.1). In what follows we assume that the constrained process is also ergodic and has for its limiting distribution the *truncation* Π^{tb} of Π to $\mathbb{M}^f$. This truncation property does not always hold, and one simple sufficient condition for it to hold is as follows (cf [15, Proposition 3.14]):

Lemma 4.1. *Suppose that q is a regular, ergodic SMQ generator, and call its unique invariant probability measure Π. Suppose that $\Pi(\mathbb{M}^f) > 0$. Suppose that the truncated generator q^{tb} is also regular and ergodic. Then the invariant probability measure Π^{tb} of the truncated process $\{N_t^{tb};\, t \geq 0\}$ is given by*

$$\Pi^{tb}(\Gamma) = \frac{\Pi(\Gamma \cap \mathbb{M}^f)}{\Pi(\mathbb{M}^f)} \tag{4.3}$$

if and only if Π satisfies the following balance equation

$$q(\nu, \mathbb{M}^f)\Pi(\mathrm{d}\nu) = \int_{\mathbb{M}^f} q(\mu, \mathrm{d}\nu)\,\Pi(\mathrm{d}\mu), \quad \nu \in \mathbb{M}^f\,.$$

The truncation property (4.3) *holds in particular if $\{N_t;\, t \geq 0\}$ is* reversible *with respect to Π, on either $\mathbb{M}^f$ or $\mathbb{M} \setminus \mathbb{M}^f$, meaning that q satisfies the following* detailed balance equation

$$q(\nu, \mathrm{d}\mu)\Pi(\mathrm{d}\nu) = q(\mu, \mathrm{d}\nu)\Pi(\mathrm{d}\mu)$$

for, either $\nu, \mu \in \mathbb{M}^f$ or $\nu, \mu \in \mathbb{M} \setminus \mathbb{M}^f$.

In what follows we assume that (4.3) holds true, in particular $\Pi(\mathbb{M}^f) > 0$, and we call $\Pi(\mathbb{M}^f)$ the *feasibility probability.* Note that $\Pi(\mathbb{M}^f)$ is the probability that the free process in steady state takes its value in the feasible part of the space.

4.1.1. *Blocking probabilities.* For given subsets $A \in \overline{\mathcal{D}}$, $B \in \mathcal{D}$, we are interested in the "ergodic frequency" p_{AB}^{tb} (to be formally defined in (4.4)) of "blocked transitions" $\nu \to T_{xy}\nu$ for $x \in A$, $y \in B$ of the process driven by q^{tb}. Actually p_{AB}^{tb} cannot be well defined given realizations of the process N_t^{tb} because "we do not observe" blocked transitions there. However, the epochs and departure-arrival locations of the *blocked transitions* can be modeled by a doubly stochastic Poisson point process $\Phi_0^{tb} = \sum_i \varepsilon_{(t_i,x_i,y_i)}$ driven by N_t^{tb}, where t_i, x_i, y_i denote, respectively, the epochs and the departure and arrival locations of the blocked transition of N_t^{tb}. Given a realization $N_\cdot^{tb} = \{N_t^{tb}, t \geq 0\}$, Φ_0^{tb} is a Poisson point process with intensity measure $\Lambda_{N_\cdot^{tb}}^{tb}$ on $(0,\infty) \times (\overline{\mathbb{D}})^2$, given by

$$\Lambda_{N_\cdot}^{tb}(D \times A \times B) = \int_D q(N_t^{tb}, T_{AB}N_t^{tb} \setminus \mathbb{M}^f)\mathrm{d}t.$$

Denote also by Φ_1^{tb} the point process on $(0,\infty)\times(\overline{\mathbb{D}})^2$ associated to ("true") transitions of N_t^{tb}, i.e.

$$\Phi_1^{tb}(D \times A \times B) = \sum_{s>0} \mathbb{1}(s \in D, N_s^{tb} = T_{xy}N_{s-}^{tb}, x \in A, y \in B).$$

Let $\Phi^{tb} = \Phi_0^{tb} + \Phi_1^{tb}$ be the superposition of Φ_i^{tb}, $i = 0, 1$. Finally define the blocking probability for the transitions $\nu \to T_{AB}(\nu)$ for some $A, B \in \overline{\mathbb{D}}$ and $\nu \in \mathbb{M}^f$ (we will call them transitions from A to B for short) as the following limit:

$$p_{AB}^{tb} = \lim_{t\to\infty} \frac{\Phi_0^{tb}((0,t] \times A \times B)}{\Phi^{tb}((0,t] \times A \times B)}\,. \tag{4.4}$$

This limit exists thanks to the following result.

Lemma 4.2. *Suppose that $\emptyset$ is a positive recurrent state for q^{tb} and denote by Π^{tb} the associated limiting distribution. If*

$$\mathbf{E}_{\Pi^{tb}}[q(N_0, \mathbb{M})] < \infty\,. \tag{4.5}$$

then

$$\lim_{t\to\infty} \frac{1}{t}\Phi_0^{tb}((0,t] \times A \times B) = \mathbf{E}_{\Pi^{tb}}[q(N_0, T_{AB}N_0 \setminus \mathbb{M}^f)]\,,$$
$$\lim_{t\to\infty} \frac{1}{t}\Phi_1^{tb}((0,t] \times A \times B) = \mathbf{E}_{\Pi^{tb}}[q(N_0, T_{AB}N_0 \cap \mathbb{M}^f)]$$

almost surely for any initial value $N_0^{tb} = \nu$ for which the return time to $\emptyset$ is finite.

Proof. Consider a probability space $(\Omega, \mathcal{F}, \mathbf{P}_\Pi)$ on which $\{N_t\}_{t\in\mathbb{R}}$ and both point processes Φ_i^{tb} $(i = 0, 1)$ are (time) stationary (see Appendix A.2). Note that the expectation corresponding to the distribution of $\{N_t\}_{t\geq 0}$ under $\mathbf{P}_\Pi$ is $\mathbf{E}_\Pi$. Condition (4.5) implies

$$\lambda_1 = \mathbf{E}_{\Pi^{tb}}[\Phi_1^{tb}((0,1] \times \overline{\mathbb{D}} \times \overline{\mathbb{D}})] = \mathbf{E}_{\Pi^{tb}}[q^{tb}(N_0^{tb})] \leq \mathbf{E}_{\Pi^{tb}}[q(N_0^{tb}, \mathbb{M})] < \infty\,,$$

where the second equality is by Lévy's formula (see Lemma A.2 in Appendix A.2). Similarly, since Φ_0^{tb} is a doubly stochastic Poisson point process,

$$\lambda_0 = \mathbf{E}_{\Pi^{tb}}[\Phi_0^{tb}((0,1] \times \overline{\mathbb{D}} \times \overline{\mathbb{D}})] = \int_0^1 \mathbf{E}_{\Pi^{tb}}[q(N_t^{tb}, \mathbb{M} \setminus \mathbb{M}^f)]\,\mathrm{d}t \leq \mathbf{E}_{\Pi^{tb}}[q(N_0^{tb}, \mathbb{M})] < \infty\,.$$

Fix $A, B \in \overline{\mathcal{D}}$. The processes $X_t^i = \Phi_i^{tb}((0,t] \times A \times B)$ $(i = 1, 2)$ are cumulative (see [1, Chapter V]) with an imbedded renewal process formed by the epochs of successive visits of N_t^{tb} to $\emptyset$. Denote by T^{tb} the generic random time between two such visits. Note that X_t^i is increasing in t and

$$\mathbf{E}_\emptyset[\sup_{0 \le t \le T^{tb}} X_t^1] = \mathbf{E}_\emptyset[X^1_{T^{tb}}] \le \lambda_1 \mathbf{E}_\emptyset[T^{tb}] < \infty\,,$$

where the inequality follows from Lemma A.1. Similarly

$$\mathbf{E}_\emptyset[\sup_{0 \le t \le T^{tb}} X_t^0] = \mathbf{E}_\emptyset[X^0_{T^{tb}}] \le \mathbf{E}_\emptyset\left[\int_0^{T^{tb}} q(N_t^{tb}, \mathbb{M} \setminus \mathbb{M}^f)\,\mathrm{d}t\right] = \lambda_0 \mathbf{E}_\emptyset[T^{tb}] < \infty\,.$$

By [1, Theorem 3.1, p. 178] we have then

$$\begin{aligned}\lim_{t\to\infty} \frac{1}{t}\Phi_1^{tb}((0,t] \times A \times B) &= \mathbf{E}_{\Pi^{tb}}[\Phi_1^{tb}((0,1] \times A \times B)] \\ &= \mathbf{E}_{\Pi^{tb}}[q(N_0^{tb}, T_{AB}N_0^{tb} \cap \mathbb{M}^f)]\,,\end{aligned}$$

where the second equality follows from Lévy's formula (see Appendix A.2). Similarly, the fact that Φ_0^{tb} is a doubly stochastic Poisson point process implies that

$$\begin{aligned}\lim_{t\to\infty} \frac{1}{t}\Phi_0^{tb}((0,t] \times A \times B) &= \mathbf{E}_{\Pi^{tb}}[\Phi_0^{tb}([0,1] \times A \times B)] \\ &= \mathbf{E}_{\Pi^{tb}}[\Lambda^{tb}_{N_\cdot}((0,1] \times A \times B)] \\ &= \mathbf{E}_{\Pi^{tb}}[q(N_0^{tb}, T_{AB}N_0^{tb} \setminus \mathbb{M}^f)]\,.\end{aligned}$$

□

The following result immediately follows from Lemma 4.2.

Proposition 4.1. *If the conditions of Lemma* 4.2 *are satisfied, then the blocking probability for transitions from A to B is*

$$p_{AB}^{tb} = \frac{\mathbf{E}_{\Pi^{tb}}[q(N_0, T_{AB}N_0 \setminus \mathbb{M}^f)]}{\mathbf{E}_{\Pi^{tb}}[q(N_0, T_{AB}N_0)]}\,.$$

Corollary 4.1. *If the conditions of Lemma* 4.2 *are satisfied, then for all $B \in \mathcal{D}$, the blocking probability for transitions from o to B is*

$$p_{\{o\}B}^{tb} = \frac{\int_B p^{tb}(o,y)\mathbf{E}_{\Pi^{tb}}[r_{oy}(N_0)]\,\lambda(o,\mathrm{d}y)}{\int_B \mathbf{E}_{\Pi^{tb}}[r_{oy}(N_0)]\,\lambda(o,\mathrm{d}y)}\,,$$

where

$$p^{tb}(o,y) = \frac{\mathbf{E}_{\Pi^{tb}}[r(N_0, T_{oy}N_0)\mathbb{1}(T_{oy}N_0 \notin \mathbb{M}^f)]}{\mathbf{E}_{\Pi^{tb}}[r(N_0, T_{oy}N_0)]}\,, \quad y \in \mathbb{D}\,. \tag{4.6}$$

If moreover (4.3) *holds, then for $B \in \mathcal{D}$,*

$$p^{tb}(o,y) = \frac{\mathbf{E}_\Pi[r(N_0, T_{oy}N_0)\mathbb{1}(N_0 \in \mathbb{M}^f, T_{oy}N_0 \notin \mathbb{M}^f)]}{\mathbf{E}_\Pi[r(N_0, T_{oy}N_0)\mathbb{1}(N_0 \in \mathbb{M}^f)]}\,, \quad y \in \mathbb{D} \tag{4.7}$$

and if $r(\nu, T_{oy}\nu) \equiv 1$ for Π-almost all $\nu \in \mathbb{M}^f$, in particular if q is a MPL generator, then we have

$$p^{tb}(o,y) = \frac{\Pi(N_0 \in \mathbb{M}^f, T_{oy}N_0 \notin \mathbb{M}^f)}{\Pi(N_0 \in \mathbb{M}^f)}\,. \tag{4.8}$$

Corollary 4.2. *If the conditions of Proposition* 3.5 *for the free generator q hold as well as Lemma* 4.2 *and Condition* (4.3)*, then*

$$p^{tb}(o,y) = \frac{\mathbf{E}_{\Pi_\rho}[r(N_0, T_{oy}N)\Psi(N_0)\mathbb{1}(N_0 \in \mathbb{M}^f, T_{oy}N_0 \notin \mathbb{M}^f)]}{\mathbf{E}_{\Pi_\rho}[r(N_0, T_{oy}N_0)\Psi(N_0)\mathbb{1}(N_0 \in \mathbb{M}^f)]}\,, \quad y \in \mathbb{D}\,. \tag{4.9}$$

and for $A, B \in \mathcal{D}$

$$p^{tb}_{AB} = \frac{\int_A \int_B p^{\mathrm{tb}}(x,y)\mathbf{E}_{\Pi_\rho}[r_{xy}\,(T_{ox}N_0)\,\mathbb{1}(T_{ox}N_0 \in \mathbb{M}^{\mathrm{f}})\Psi(T_{ox}N_0)]\rho\,(dx)\,\lambda\,(x,dy)}{\int_A \int_B \mathbf{E}_{\Pi_\rho}[r_{xy}\,(T_{ox}N_0)\,\mathbb{1}(T_{ox}N_0 \in \mathbb{M}^{\mathrm{f}})\Psi(T_{ox}N_0)]\rho\,(dx)\,\lambda\,(x,dy)}\,,$$

where

$$p^{tb}(x,y) = \frac{\mathbf{E}_{\Pi_\rho}[r(N_0, T_{oy}N_0)\Psi(T_{ox}N_0)\mathbb{1}(T_{ox}N_0 \in \mathbb{M}^f, T_{oy}N_0 \notin \mathbb{M}^f)]}{\mathbf{E}_{\Pi_\rho}[r(N_0, T_{oy}N_0)\Psi(T_{ox}N_0)\mathbb{1}(T_{ox} \in \mathbb{M}^f)]}\,, \quad x \in \mathbb{D}, y \in \overline{\overline{\mathbb{D}}}. \tag{4.10}$$

If $r(\nu, T_{oy}\nu) \equiv 1$ for Π_ρ-almost all $\nu \in \mathbb{M}^f$, in particular if q is a MPL generator, then we have

$$p^{tb}(x,y) = \frac{\Pi_\rho(T_{ox}N_0 \in \mathbb{M}^f, T_{oy}N_0 \notin \mathbb{M}^f)}{\Pi_\rho(T_{ox}N_0 \in \mathbb{M}^f)}\,, \quad x \in \mathbb{D}, y \in \overline{\overline{\mathbb{D}}}. \tag{4.11}$$

We will call $p^{tb}(x,y)$ the *blocking rate* from x to y.

Remark 4.1. Note that formula (4.8) might be seen as a *spatial extension of the classical Erlang formula*.

4.2. Blocking versus cut probabilities

In this section we will consider the forced termination model and will clearly distinguish between *blocking of new arrivals* and *cutting of existing calls in progress*.

We suppose that q^{ft} is regular and ergodic, and that it has an invariant distribution that we denote by Π^{ft}. Note that Π^{ft} is a probability distribution on $\mathbb{M}$.

4.2.1. *Blocking probabilities.* As in the previous section, the *blocked arrivals* can be modeled by a doubly stochastic Poisson point process $\Phi_0^{ft} = \sum_i \varepsilon_{(t_i y_i)}$ driven by N_t^{ft}, where t_i, y_i denote, respectively, the epochs and arrival locations of blocked transition of N_t^{ft}. Given a realization $N_\cdot^{ft} = \{N_t^{ft}, t \geq 0\}$, Φ_0^{ft} is a Poisson point process with intensity measure $\Lambda^{ft}_{N_\cdot^{ft}}$ on $(0,\infty) \times \mathbb{D}$, given by

$$\Lambda^{ft}_{N_\cdot}(D \times B) = \int_D q(N_t^{ft}, T_{oB}N_t^{ft} \setminus \mathbb{M}^f)\mathrm{d}t.$$

Denote also by Φ_1^{tb} the point process on $(0,\infty) \times \mathbb{D}$ associated to ("true") arrivals of N_t^{ft}, i.e.

$$\Phi_1^{ft}(D \times B) = \sum_{s>0} \mathbb{1}(s \in D, N_s^{ft} = T_{oy}N_{s-}^{ft}, y \in B).$$

Let $\Phi^{ft} = \Phi_0^{ft} + \Phi_1^{ft}$ be the superposition of Φ_i^{ft}, $i = 0, 1$. Finally define the blocking probability for the transitions $\nu \to T_{oB}(\nu)$ for some $B \in \mathbb{D}$ and $\nu \in \mathbb{M}^f$ (will call them arrivals to B for short) as the following limiting ratio of blocked transitions to all transitions

$$p_{oB}^{tb} = \lim_{t\to\infty} \frac{\Phi_0^{ft}((0,t] \times B)}{\Phi^{ft}((0,t] \times B)} .$$

The following result can be proved along the same lines as Lemma 4.2.

Proposition 4.2. *Suppose that $\emptyset$ is a positive recurrent state for q^{ft} with the limiting distribution Π^{ft}. If*

$$\mathbf{E}_{\Pi^{ft}}[q(N_0, \mathbb{M})] < \infty \tag{4.12}$$

then

$$\lim_{t\to\infty} \frac{1}{t}\Phi_0^{ft}((0,t] \times B) = \mathbf{E}_{\Pi^{ft}}[q(N_0, T_{oB}N_0 \setminus \mathbb{M}^f)] ,$$
$$\lim_{t\to\infty} \frac{1}{t}\Phi_1^{ft}((0,t] \times B) = \mathbf{E}_{\Pi^{ft}}[q(N_0, T_{oB}N_0 \cap \mathbb{M}^f)]$$

almost surely for any initial value $N_0^{ft} = \nu$ for which the return time to $\emptyset$ is finite. Moreover

$$p^{ft}{}_{oB} = \frac{\mathbf{E}_{\Pi^{ft}}[q(N_0, T_{oB}N_0 \setminus \mathbb{M}^f)]}{\mathbf{E}_{\Pi^{ft}}[q(N_0, T_{oB}N_0)]} = \frac{\int_B p^{ft}(o,y)\mathbf{E}_{\Pi^{ft}}[r_{oy}(N_0)]\,\lambda(o, \mathrm{d}y)}{\int_B \mathbf{E}_{\Pi^{ft}}[r_{oy}(N_0)]\,\lambda(o, \mathrm{d}y)} ,$$

where

$$p^{ft}(o,y) = \frac{\mathbf{E}_{\Pi^{ft}}[r(N_0, T_{oy}N_0)\mathbb{1}(T_{oy}N_0 \notin \mathbb{M}^f)]}{\mathbf{E}_{\Pi^{ft}}[r(N_0, T_{oy}N_0)]} \quad y \in \mathbb{D} . \tag{4.13}$$

4.2.2. *Cut probabilities* We are now interested in forced terminations (cuts) of the service. Looking at the form of the generator q^{ft} we see that each transition $\nu \to T_{xo}\nu$ for some $x \in \nu$ can be, independently of everything else, either a "regular termination" with probability

$$\tau_{\nu,x}(\{o\}) = \frac{r(\nu, T_{xo}\nu)}{r^{ft}(\nu, T_{xo}\nu)}$$

or a "forced termination" due to an unsuccessful displacement to $B \in \mathbb{D}$ with probability

$$\tau_{\nu,x}(B) = \frac{\int_B r(\nu, T_{xy}\nu)\mathbb{1}(T_{xy}\nu \notin \mathbb{M}^f)\,\lambda(x, \mathrm{d}y)}{r^{ft}(\nu, T_{xo}\nu)\lambda(x,o)} .$$

So we can model different terminations of the process N^{ft} by a marked point process $\Phi_2^{ft} = \sum_i \varepsilon_{(t_i,x_i,y_i)}$ where t_i are termination epochs, $x_i \in \mathbb{D}$ denote the departure locations and $y_i \in \overline{\mathbb{D}}$ denote the termination status: $y_i = o$ if it is a regular one and $y_i \in \mathbb{D}$ if it is caused by an unsuccessful displacement from x_i to y_i. Note that given a realization of N_t^{ft}, the points t_i and marks x_i are known, and we assume that y_i are independently chosen with the distribution $\tau_{N_{t_i}^{ft},x_i}(B)$. Considering marked point processes of epochs and departure-arrival location of all transition, we can express various ergodic limit fractions of users cut on when trying to move from some given

$A \in \mathcal{D}$ to $B \in \mathcal{D}$. Here we will show only how to derive ergodic limit fractions c of users that are forced to terminate during their sojourn in the network:

$$c = \lim_{t\to\infty} \frac{\Phi_2^{ft}((0,t] \times \mathbb{D} \times \mathbb{D})}{\Phi_1^{ft}((0,t] \times \mathbb{D})}\,.$$

Proposition 4.3. *Suppose that $\emptyset$ is a positive recurrent state for q^{ft} with the limiting distribution Π^{ft}. If*

$$\mathbf{E}_{\Pi^{ft}}[q(N_0, T_{o\mathbb{D}}N_0 \cap \mathbb{M}^f)] < \infty \tag{4.14}$$

and

$$\mathbf{E}_{\Pi^{ft}}[q(N_0, T_{\mathbb{D}\mathbb{D}}N_0 \setminus \mathbb{M}^f)] < \infty \tag{4.15}$$

then the limit c exists almost surely for any initial value $N_0^{ft} = \nu$ for which the return time to $\emptyset$ is finite and

$$c = \frac{\mathbf{E}_{\Pi^{ft}}[q(N_0, T_{\mathbb{D}\mathbb{D}}N_0 \setminus \mathbb{M}^f)]}{\mathbf{E}_{\Pi^{ft}}[q(N_0, T_{o\mathbb{D}}N_0 \cap \mathbb{M}^f)]}\,. \tag{4.16}$$

where

Proof. Note the processes $X_t = \Phi_2^{ft}([0,t] \times \mathbb{D} \times \mathbb{D})$ is cumulative with the imbedded renewal process being the epochs of successive visits of N_t^{ft} at $\emptyset$. Following the same lines as in the proof of Lemma 4.2 we find under condition (4.15) that $\lim_{t\to\infty} 1/t\ \Phi_2^{ft}((0,t] \times \mathbb{D}^2) = \mathbf{E}_\Pi^{ft}[\Phi_2^{ft}((0,1] \times \mathbb{D}^2)]$. Similarly, under condition (4.14) $\lim_{t\to\infty} 1/t\ \Phi_1^{ft}((0,t] \times \mathbb{D}) = \mathbf{E}_\Pi^{ft}[\Phi_1^{ft}((0,1] \times \mathbb{D})]$. By Lévy's formula we obtain (4.16).

5. Applications

5.1. Blocking and cuts in large CDMA networks

The Code Division Multiple Access (CDMA) is a medium access technique used in modern cellular telephony networks (as e.g. UMTS). In CDMA, a given configuration of channels with predefined bit-rates is feasible if there exists some vector of emitted powers which guarantee that the Signal-to-Interference-and-Noise-Ratio (SINR) at each receiver exceeds the threshold required for the bit-rate of the associated channel. The solution to this power allocation problem, which might be constrained by further limitations on the maximum power of stations/users, is a complex problem especially when the numbers of users and stations are large. In [6, 3] some sufficient conditions for feasibility of this problem have been proposed. They allow to characterize spatial patterns of users that can be entirely accepted by the network, given required communication rates. This makes it possible to model and analyze a large CDMA network by a spatial version of the classical loss system, where each serving Base Station (BS) has some capacity, say C, and each user served by the station brings its load, which depends on the location of the user in the space (more precisely on its distance to the serving station and to other interfering stations) and on the channel communication rate. The general feasibility condition (for the whole network) consists of a set of inequalities, one per each station, postulating that the sum of loads of users served by each station should not be larger than C. In the case of a spatially periodic network of stations (see balow), this decentralized form allows one to analyze blocking and cuts of users by considering their configurations separately in each cell of the network. We will explain this idea in the remaining part of this section by considering just one (typical) cell of the infinite network.

In order to calculate the feasibility and blocking probabilities, we have to fix a network architecture and its parameters, as well as the free process of arrivals.

5.1.1. *Infinite hexagonal network of BS's.* In what follows, we consider the spatially periodic pattern of base stations often called the *honeycomb* model, where the BS's are located on the following grid in the complex plane:

$$\{Y^u : Y^u = \Delta(u_1 + u_2 e^{i\pi/3}),\ u = (u_1, u_2) \in \{0, \pm 1, \dots\}^2\}.$$

Note that we consider the honeycomb on the whole plane. We will denote by Y^u the location of BS u. We denote by λ_{BS} the mean number of BS's per unit of space. Each BS serves users in its *cell* defined as the set of locations in the plane which are closer to that BS than to any other BS. Each cell is hexagonal. It is convenient to relate λ_{BS} to the radius R of the (virtual) disc whose area is equal to that of the cell, by the formula

$$\lambda_{BS} = 1/(\pi R^2)\,.$$

With this definition in mind, we will sometimes call R the *radius of the cell.* In this hexagonal model, the radius R is related to the distance Δ between two adjacent BS's by the formula $\Delta^2 = 2\pi R^2/\sqrt{3}$.

5.1.2. *Path-loss.* We model path-loss on distance r by

$$L(r) = (Kr)^\eta, \tag{5.1}$$

where $\eta > 2$ is the so-called *path-loss* exponent and $K > 0$ is a multiplicative constant.

5.1.3. *Free process of calls.* Fix one cell of the honeycomb described above, say that located at the origin $Y^0 = 0$; for simplicity we will omit the superscript 0 in what follows. Following the notation of the previous sections, we denote this cell, considered as a subset of $\mathbb{R}^2$, by $\mathbb{D}$. We will model the call arrivals mobility and departures from $\mathbb{D}$ by a MPL process (see Example 3.1). We will also consider a simplified model, in which mobility of calls is not taken into accout, taking a SBD process (see Example 3.2) as free process. Note that in both cases, for a given $A \in \mathcal{D}$, interarrival times to A are independent exponential random variables with mean $1/\lambda(o, A)$, where $\lambda(o, \cdot)$ is some given intensity measure of arrivals to $\mathbb{D}$ per unit of time. This allows the modeling of spatial hot spots. In homogeneous traffic conditions, we can take $\lambda(o, \mathrm{d}x) = \lambda \mathrm{d}x$, where λ is the mean number of arrivals per unit of area and per unit of time. We assume that call holding times are independent exponential random variables with mean $1/\tau$. This description corresponds to the assumption $\lambda(x, o) = \tau$. (Note that in both models, MPL and SBD, $r(\nu, T_{xy}\nu) = 1$ for $x, y \in \overline{\mathbb{D}}$). The call mobility inside the cell is represented by some given routing kernel $\lambda(x, \mathrm{d}y)$ $(x, y \in \mathbb{D})$, and in the case of SBD process, it is taken to be null $(\lambda(x, \mathbb{D}) = 0)$. It is not difficult to check that if $\sup_{x\in\overline{\mathbb{D}}} \lambda(x, \mathbb{D}) < \infty$ and $\tau > 0$, the associated generator q is regular and ergodic for both the MPL and the SBD case. By Proposition 3.5 (in fact by Remark 3.5 in the case of a non reversible mobility kernel), the invariant measure Π_ρ is that of a Poisson point process with mean measure ρ being the solution to (3.17). In the case of a SBD process, $\rho(\mathrm{d}x) = \lambda(o, \mathrm{d}x)/\tau$.

5.1.4. *Feasibility condition.* We remind that here $\nu \in \mathbb{M}$ is a counting measure that characterizes the configuration of user locations in cell $\mathbb{D}$. The admission/congestion control protocols described in [3], consists in the application of the following rule by the BS: it only accepts user configurations ν that satisfy some conditions, whose general form is

$$\int \tilde{f}(x)\,\nu(\mathrm{d}x) < C \tag{5.2}$$

for some constant C and a non-negative function $\tilde{f}$ defined on $\mathbb{D}$ which is of the form

$$f(x) = \sum_{v \neq 0} \frac{L(|x|)}{L(|Y^v - x|)},$$

where v ranges over the set of base stations. We define the set of feasible configurations $\mathbb{M}^f$ as all configurations $\nu \in \mathbb{M}$ that satisfy condition (5.2). The feasibility probability is equal to $\Pi_\rho(\mathbb{M}^f) = \Pi_\rho\{\int_{\mathbb{D}} \tilde{f}(x)\,N(\mathrm{d}x) < C\}$ and it is shown in [3] how it can be estimated via Gaussian approximations (note that under Π_ρ, the sum $\int_{\mathbb{D}} \tilde{f}(x)\,N(\mathrm{d}x)$ is a compound Poisson random variable with known mean and variance).

We will briefly discuss now the pertinence of the two loss models proposed in Section 4 in conjunction with the above free process of calls and feasibility condition.

5.1.5. *Call blocking with "pedestrian" mobility.* Note that in the transition blocking model driven by the modification q^{tb} of the free generator q, there are no losses of calls in progress: an unauthorized displacement is blocked and the call in question remains at its previous location until the next event. This assumption might be judged realistic in the absence of mobility (i.e. with SBD free process) or in the case where a call loosing its connection is able to instantaneously "return" to its previous position where the connection was assured. We call it a "pedestrian" mobility scenario. It is not difficult to verify that the conditions of Corollary 4.2 are satisfied and thus

$$p^{tb}(o,y) = \frac{\Pi_\rho\{\, C - \tilde{f}(y) \le \int_{\mathbb{D}} \tilde{f}(z)\,N(\mathrm{d}z) < C \,\}}{\Pi_\rho\{\, \int_{\mathbb{D}} \tilde{f}(z)\,N(\mathrm{d}z) < C \,\}} \quad y \in \mathbb{D}\,, \tag{5.3}$$

$$p^{tb}(x,y) = \frac{\Pi_\rho\{\, C - \tilde{f}(y) \le \int_{\mathbb{D}} \tilde{f}(z)\,N(\mathrm{d}z) < C - f(x) \,\}}{\Pi_\rho\{\, \int_{\mathbb{D}} \tilde{f}(z)\,N(\mathrm{d}z) < C - f(x) \,\}} \quad x, y \in \mathbb{D} \tag{5.4}$$

are respectively the arrival and displacement blocking rates. Formula (5.3) was proven in [4] in the context of a SBD free process. The Gaussian approximations of the feasibility probability in the denominator of (5.3) and analogous approximations for the numerator lead to explicit approximate expressions for the blocking rates in this model. For example (5.3) can be approximated by the expression

$$p^{tb}(o,y) \approx \frac{Q\big((C - \tilde{f}(y) - \mu)/\sigma\big) - Q\big((C-\mu)/\sigma\big)}{1 - Q\big((C-\mu)/\sigma\big)}\,, \tag{5.5}$$

where $Q(z) = 1/\sqrt{2\pi}\int_z^\infty e^{-t^2/2}\,\mathrm{d}t$ is the Gaussian tail distribution function and μ, σ^2 denote, respectively, the mean and the variance of $\int_{\mathbb{D}} \tilde{f}(z)\,N(\mathrm{d}z)$ under Π_ρ. In order to calculate these constants, one can use the following approximation for $f(x)$ proposed in [12]

$$f(x) \approx \zeta(\eta-1)L(|x|) \times \left(\frac{1}{L(\Delta-|x|)} + \frac{1}{L(\Delta+|x|)} + \frac{4}{L\left(\sqrt{\Delta^2+|x|^2}\right)}\right),$$

for $|x| \leq R$, where $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$ is the Riemann zeta function (recall that Δ is the distance between two adjacent BS's in the hexagonal network and R is the radius of the disc with area equal to that of the cell). To be more specific, for one of the admission/congestion control protocols considered in [4], we have $\tilde{f}(x) = \xi'_{\downarrow}(\alpha + f(x))$, where $\xi'_{\downarrow}$ and α are some constants related, respectively, to the channel bit-rate and the coding technique, and $C = 1$. In this case, under some homogeneous traffic scenario, $\bar{M} = \lambda/(\tau\pi R^2)$ is the mean number of users of the free process in the cell and $\mu = \bar{M}\xi'_{\downarrow}(\alpha + \bar{f})$ and $\sigma^2 = \bar{M}{\xi'_{\downarrow}}^2(\alpha^2 + 2\alpha\bar{f} + \overline{f^2})$ where $\bar{f} \approx 0.9365/(\eta-2)$ and $\overline{f^2} \approx 0.2343/(\eta-2) + 1.2907/(\eta-2)^2$. We refer to [4] for more examples.

5.1.6. *Blocking and cut probabilities.* When mobility cannot be ignored and when we cannot assume that calls loosing their connection are able to instantaneously "return" to their previous positions, we have to use the forced termination model. It is also not difficult to verify that the modification q^{ft} of our free generator q is regular and ergodic and that the conditions of Proposition 4.3 are satisfied. However the invariant measure Π^{ft} is not known explicitly. Thus Formula (4.16) has to be further investigated. We mention here a few ideas, whose development is beyond the scope of this paper.

- Numerical estimation by the (perfect) simulation of the realizations of N under Π^{ft}.
- Exact numerical calculations (via discretization).
- Bounds and limiting approximations (for example when the mean mobility tends to 0 or of the mean-field type).

5.2. Hard-core blocking in CSMA

Carrier Sense Multiple Access (CSMA) is a medium access protocol for wireless communications where each communication within a targeted transmission range R is protected by some exclusion disc centered at the transmitter with radius $R_{cs} > R$. Modeling the arrivals mobility and departures of transmission demands by some MPL or SBD process (as in Section 5.1) and defining the set of feasible configurations as $\mathbb{M}^f = \{\nu \in \mathbb{M} : \forall_{x,y\in\nu} |x-y| > R_{cs}\}$, we can study the temporal evolution of an ad-hoc network implementing such a CSMA protocol. In particular, in the case of the blocking transition loss model based on some reversible free process, the invariant measure Π^{tb} will be a Gibbs hard-core measure.

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Appendix

The results gathered in this section are supposed to make the paper more self-contained.

A.1. Discrete birth-and-death process

A discrete birth-and-death process is a Markov process on the state space $\mathbb{N} = \{0, 1, \ldots\}$ that only jumps up or down by 1. Its infinitesimal generator is thus of the form

$$\begin{aligned} q'_{n,n+1} &= b_n\,, \quad n \geq 0\,, \\ q'_{n,n-1} &= d_n\,, \\ q'_n &= b_n + d_n\,. \end{aligned} \tag{A.1}$$

with $0 \leq b_n, d_n < \infty$ and $d_0 = 0$. The following sufficient conditions for a birth-and-death generator q' to be regular are given in [14]. The generator q' is regular if either of the two conditions is satisfied for some $n_0 \geq 0$:

$$b_n = 0, \quad \forall n \geq n_0 \tag{A.2}$$

or

$$b_n > 0, \quad \forall n \geq n_0 \text{ and } \sum_{n=n_0}^{\infty} w_n = \infty \tag{A.3}$$

where

$$w_n = \frac{1}{b_n} + \frac{d_n}{b_n b_{n-1}} + \ldots + \frac{d_n \ldots d_{n_0+1}}{b_n \ldots b_{n_0}} + \frac{d_n \ldots d_{n_0}}{b_n \ldots b_{n_0}}\,.$$

Note that (A.3) is satisfied if

$$b_n > 0, \quad \forall n \geq n_0 \text{ and } \sum_{n=n_0}^{\infty} \frac{1}{b_n} = \infty$$

or

$$b_n > 0, \quad \forall n \geq n_0 \text{ and } \sum_{n=n_0}^{\infty} \frac{d_n \ldots d_{n_0}}{b_n \ldots b_{n_0}} = \infty$$

In particular q' is regular if $\sup_{n\geq 0} b_n < \infty$. The following conditions related to ergodicity of q' are given in [11]. The state 0 a positive recurrent state of q' if and only if one the following holds:

$$b_n = 0, \quad \text{for some } n \geq 1 \quad \text{and } d_m > 0 \quad \forall 0 < m \leq n\,, \tag{A.4}$$

or

$$b_n > 0 \quad \forall n \geq 0,\, d_n > 0 \quad \forall n > 0 \,\text{and} \quad \sum_{n=1}^{\infty} \frac{b_0 \ldots b_{n-1}}{d_1 \ldots d_n} < \infty\,. \tag{A.5}$$

A.2. Markov jump process and associated point processes

In Section 4 we consider a few (marked) point processes on the real line, which are all generated is some way by our SMQ process $\{N_t\}$ with the regular and ergodic generator q. Here we remind briefly the basic relations between the stationary and Palm distributions associated to these point process. Suppose that the assumptions of the Proposition 3.3 hold and let Π be the invariant measure given by (3.11). We consider a measurable space $(\Omega, \mathcal{F})$ equipped with the measurable flow $\{\theta_t\}_{t\in\mathbb{R}}$ (see [9, Section 1.1.2]) on which $\{N_t\}_{t\in\mathbb{R}}$ and all the considered point processes are defined. In particular in what follows we denote by $\Phi_\emptyset$ the point process that counts the visits (entrance times) of $\{N_t\}_{t\in\mathbb{R}}$ to $\emptyset$ and by and Φ the one that counts all the jumps of $\{N_t\}_{t\in\mathbb{R}}$. One can consider the following three probability measures on $(\Omega, \mathcal{F}, \theta_t)$ (the third one is not used in the sequel):

- $\mathbf{P}^0_\emptyset$ is the measure under which at time 0 there is a jump of $\{N_t\}$ which brings the process to the value $N_0 = \emptyset$. We may consider it as the Palm measure associated to $\Phi_\emptyset$. Expectation under this measure is denoted by $\mathbf{E}_\emptyset$ in the main stream of this article.
- $\mathbf{P}_\Pi$ is the stationary probability measure, which may be obtained from $\mathbf{P}^0_\emptyset$ by the Slivnyak inverse construction (see [9, Section 1.3.5]).
- $\mathbf{P}^0$ is the Palm measure associated to Φ. It can be constructed from $\mathbf{P}_\Pi$ provided $\mathbf{E}_\Pi[\Phi(0,1]] = \mathbf{E}_\Pi[q(N)] < \infty$ (for the equality see Lemma A.2 below). It is known that under $\mathbf{P}^0$ the value N_0 (taken by the process just after the jump made at 0) is distributed according to $q(N)\Pi(\mathrm{d}N)/\mathbf{E}_\Pi[q(N)]$.

Denote by $T_n^\emptyset$ the points of $\Phi_\emptyset$ (note that in the main stream of the paper $T_1^\emptyset \equiv T$). Here is a result that follows from the Swiss army formula (see [9, Section 1.3.7]).

Lemma A.1. *Let Φ' be any point process on $(\Omega, \mathcal{F}, \theta_t)$. Then under the foregoing assumptions,*

$$\mathbf{E}_\Pi[\Phi'(0,1]] = \frac{\mathbf{E}^0_\emptyset[\Phi'(0, T_1^\emptyset]]}{\mathbf{E}^0_\emptyset[T_1^\emptyset]}. \tag{A.6}$$

Suppose now that Φ_H is a point process counting the H-transitions of $\{N_t\}$, where H is a measurable subset of $\{(\nu,\mu) \in \mathbb{M}^2 : \nu \neq \mu\}$; i.e., $\Phi_H(B) = \int_B \mathbb{1}((N_{t-}, N_t) \in H)\,\Phi(\mathrm{d}t)$. We have the following version of Lévy's formula for Φ_H.

Lemma A.2. *Under the foregoing assumptions, the point process Φ_H counting the H-transitions of $\{N_t\}$ satisfies the relation*

$$\mathbf{E}_\Pi[\Phi_H(0,1]] = E_\Pi[q(N, H_N)]\,, \tag{A.7}$$

where $H_N = \{\nu \in \mathbb{M} : (N,\nu) \in H\}$.

Proof. Denote by $\{N_k\}$ the discrete imbedded Markov chain of $\{N_t\}$ ($N_k = N_{T_k}$, where $\{T_k\}$ are the jump epochs of $\{N_t\}$). Let $Z_k = \mathbb{1}((N_{k-1}, N_k) \in H)$ and denote by $\lambda_\emptyset = 1/\mathbf{E}^0_\emptyset[T_1^\emptyset]$ the intensity of $\Phi_\emptyset$. By (A.6)

$$\begin{aligned}
\mathbf{E}_\Pi[\Phi_H(0,1]] &= \lambda_\emptyset \mathbf{E}^0_\emptyset\Big[\sum_{k\geq 1} Z_k \mathbb{1}(T_k \leq T_1^\emptyset)\Big] \\
&= \lambda_\emptyset \mathbf{E}^0_\emptyset\Big[\sum_{k\geq 1} Z_k \mathbb{1}(N_1 \neq \emptyset, \ldots, N_{k-1} \neq \emptyset) q(N_{k-1}) \mathbf{E}^0_\emptyset[T_k - T_{k-1} | \{N_n\}]\Big] \\
&= \lambda_\emptyset \mathbf{E}^0_\emptyset\Big[\sum_{k\geq 1} Z_k \mathbb{1}(N_1 \neq \emptyset, \ldots, N_{k-1} \neq \emptyset) q(N_{k-1})(T_k - T_{k-1})\Big] \\
&= \lambda_\emptyset \mathbf{E}^0_\emptyset\Big[\int_0^{T_1^\emptyset} Z_0 \circ \theta_t q(N_{-1} \circ \theta_t)\,\mathrm{d}t\Big] \\
&= \mathbf{E}_\Pi[Z_0 q(N_{-1})]\,,
\end{aligned}$$

where the last equality is due to the inversion formula [9, Formula (1.2.25) p.20]. Note now that

$$\begin{aligned}
\mathbf{E}_\Pi[Z_0 q(N_{-1})] &= \mathbf{E}_\Pi[q(N_{-1})\mathbb{1}((N_{-1}, N_0) \in H)] \\
&= \mathbf{E}_\Pi\Big[\int_{\mathbb{M}} \mathbb{1}(1(N_{-1}, \nu) \in H) q(N_{-1}, \mathrm{d}\nu)\Big] \\
&= \mathbf{E}_\Pi[q(N_{-1}, H_{N_{-1}})],
\end{aligned}$$

which completes the proof.

□

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